

Example 1.

Diagonalized the matrix by orthogonal transformation

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Sol

Find characteristic equation, eigenvalue & eigenvector

Characteristic equation

$$\lambda^3 - c_1 \lambda^2 + c_2 \lambda - c_3 = 0$$

$c_1 =$  Sum of the main diagonal elements

$$= 8 + 7 + 3 = 18$$

$c_2 =$  Sum of minor of main diagonal elements

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= (21 - 16) + (24 - 4) + (56 - 36)$$

$$= 5 + 20 + 20 = 45$$

$$c_3 = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + 6 \begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}$$

$$= 8(21 - 16) + 6(-18 + 8) + 2(24 - 14)$$

$$= 8(5) + 6(-10) + 2(10)$$

$$= 40 - 60 + 20 = 0$$

C.E

$$\lambda^3 - 18\lambda^2 + 45\lambda + 0 = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 15$$

$$\begin{array}{c} 45 \\ \wedge \\ -3 \quad -15 \end{array}$$

To find eigen vector

The eigen vector  $X$  corresponding to the eigen value  $\lambda$  is given by  $(A - \lambda I)X = 0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case i)  $\lambda = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{ccc} 8 & -6 & 2 \\ -6 & 7 & -4 \end{array}$$

Row must be different

$$\begin{array}{cccc} & x_1 & & \\ 8 & -6 & 2 & 8 \\ -6 & 7 & -4 & -6 \\ \hline x_3 & & x_2 & \end{array}$$

Repeat first column

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 8 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x_1}{24 - 14} = \frac{x_2}{-12 + 32} = \frac{x_3}{56 - 36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case ii)  $\lambda = 3$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{ccc} 5 & -6 & 2 \\ -6 & 4 & -4 \\ \hline 5 & -6 & 2 & 5 \\ -6 & 4 & -4 & -6 \\ \hline x_3 & & x_2 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 5 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$x_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Case iii)  $\lambda = 15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{ccc} -7 & -6 & 2 \\ -6 & -8 & -4 \end{array}$$

$$\begin{array}{cccc} & & x_1 & \\ -7 & -6 & 2 & -7 \\ \hline & -6 & -8 & -4 & -6 \\ & x_3 & & x_2 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & -7 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Step 2

check orthogonal condition

T.P  $x_1^T x_2 = 0$

$$x_1^T x_2 = [1 \ 2 \ 2] \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$= 2 + 2 - 4 = 0$$

$$x_2^T x_3 = [2 \ 1 \ -2] \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$= 4 - 2 - 2$$

$$= 0$$

$$x_3^T x_1 = [2 \ -2 \ 1] \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= 2 - 4 + 2$$

$$= 0$$

Step 3

Find normalized matrix

Eigen vector

$$l(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Normalized vector

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$l(x_1) = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$$

$$N_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$l(x_2)$$

$$N_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$N_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

Step 4

Modal matrix

$$N = [N_1 \quad N_2 \quad N_3]$$

$$= \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$N^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$N^T A N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} = D \quad \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$