SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

GRADIENT, DIVERGENCE & CURL

Vector differential operator: ∇

- $\nabla \equiv \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k} = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$

- The operator ∇ is known as ‘***Nabla***’

The Gradient: $\nabla\phi$

- Let $\phi(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space. Then ***the gradient of*** ϕ, is defined by

$$\nabla\phi \equiv \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right)\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$$

The Divergence: $\nabla . \mathrm{V}$

- Let $\boldsymbol{V} = V_1\boldsymbol{i} + V_2\boldsymbol{j} + V_3\boldsymbol{k}$ be defined and differentiable at each point (x, y, z) in a certain region of space.
- Then ***the divergence of*** $\boldsymbol{V}$, is defined by

$$\nabla \cdot \mathrm{V} \equiv \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot (V_1 i + V_2 j + V_3 k) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

N.B.: $\boldsymbol{\nabla} \cdot \boldsymbol{V}$ defines a scalar Field

The Curl: $\nabla \times \boldsymbol{V}$

- Let $\boldsymbol{V} = V_1\boldsymbol{i} + V_2\boldsymbol{j} + V_3\boldsymbol{k}$ be defined and differentiable at each point (x, y, z) in a certain region of space. Then ***the curl of*** $\boldsymbol{V}$, is defined by

$$\nabla \times \mathbf{V} \equiv \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \times (V_1\boldsymbol{i} + V_2\boldsymbol{j} + V_3\boldsymbol{k})$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

N.B.: $\nabla \times \boldsymbol{V}$ defines a vector field

Some properties

1. $\nabla \times \mathbf{V} \equiv \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\boldsymbol{k}$
2. $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$
3. $\nabla \cdot (\phi \mathbf{A}) = (\nabla \phi) \cdot \mathbf{A} + \phi(\nabla \cdot \mathbf{A})$
4. $\nabla \times (\phi \mathbf{A}) = (\nabla \phi) \times \mathbf{A} + \phi(\nabla \times \mathbf{A})$
5. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
6. $\nabla \cdot (\nabla \phi) \equiv \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$
7. $\nabla \times (\nabla \cdot A) \equiv \mathbf{0}$
8. $\nabla \cdot (\nabla \times A) \equiv \mathbf{0}$

Problem 1: If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\phi$ at the point (1, -2,-1).

Solution: $\nabla\phi = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right)(3x^2y - y^3z^2)$

$$= \boldsymbol{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \boldsymbol{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) +$$

$$\boldsymbol{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= 6xy\ \ \boldsymbol{i} + (3x^2 - 3y^2z^2)\ \boldsymbol{j} - 2y^3z\ \ \boldsymbol{k}$$

$$\nabla\phi\Big|_{(1,-2,-1)} = 6(1)(-2)\,\boldsymbol{i} + \{3(1)^2 - 3(-2)^2(-1)^2\}\,\boldsymbol{j}$$

$$-2(-2)^3(-1)\,\boldsymbol{k}$$

$$= -12\,\boldsymbol{i} - 9\,\boldsymbol{j} - 16\ \boldsymbol{k}$$

Problem 2: If $\boldsymbol{A} = x^2z\,\boldsymbol{i} - 2y^3z^2\boldsymbol{j} + xy^2z\,\boldsymbol{k}$, find $\nabla \cdot \boldsymbol{A}$ at the point (1,-1,1).

Solution:

$$\nabla \cdot \boldsymbol{A} = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \cdot (x^2z\,\boldsymbol{i} - 2y^3z^2\boldsymbol{j} + xy^2z\,\boldsymbol{k})$$

$$= \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz - 6y^2z^2 + xy^2$$

$$\nabla \cdot \boldsymbol{A}\Big|_{(\boldsymbol{1},-\boldsymbol{1},\boldsymbol{1})} = 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2$$

$$= -3$$

Problem 3: If $\phi = 2x^3y^2z^4$, find $\nabla \cdot \nabla\phi$.

Solution: $\nabla\phi = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right)(2x^3y^2z^4)$

$$= \boldsymbol{i}\frac{\partial}{\partial x}(2x^3y^2z^4) + \boldsymbol{j}\frac{\partial}{\partial y}(2x^3y^2z^4) + \boldsymbol{k}\frac{\partial}{\partial z}(2x^3y^2z^4)$$

$$= 6x^2y^2z\,\boldsymbol{i} + 4x^3yz^4\boldsymbol{j} + 8x^3y^2z^3\,\boldsymbol{k}$$

Then,

$$\nabla \cdot \nabla\phi = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \cdot (6x^2y^2z\boldsymbol{i} + 4x^3yz^4\boldsymbol{j} + 8x^3y^2z^3\,\boldsymbol{k})$$

$$= \frac{\partial}{\partial x}(6x^2y^2z) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

THANK YOU