Q/ An elastic membrane when x_1, x_2, plane boundary circles $x_1^2 + x_2^2 = 1$ is stretched so that the point $P : (x_1, x_2)$ goes over into the point $Q : (y_1, y_2)$ given by

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ and } \lambda x = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ in}$$

components $y_1 = 5x_1 + 3x_2$

$y_2 = 3x_1 + 5x_2$

find the principle direction, that is the direction of position vector x of P for which the direction of position vector y of Q is the same (or) exactly opposite. what shape does the boundary circle take under this deformation.

Soln:- We know that $y = Ax$

Now, $y = \lambda x$

comparing $Ax = \lambda x$

$(A - \lambda I)x = 0$

The characteristic equation $\lambda^2 - D_1\lambda + D_2 = 0$

$D_1 = 5 + 5 = 10$

$D_2 = 25 - 9 = 16$

$\therefore \lambda^2 - 10\lambda + 16 = 0$

$\lambda = 8, 2$

Eigen values are 2 and 8

Eigen vectors : $(A - \lambda I)x = 0$

$$\begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

when $\lambda = 2$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + 3x_2 = 0$$
$$3x_1 + 3x_2 = 0$$

eq'n are same so $3x_1 + 3x_2 = 0$

$$x_1 = -x_2$$

$$\frac{x_1}{-1} = \frac{x_2}{1}$$

$$\therefore X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

when $\lambda = 8$

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + 3x_2 = 0$$
$$3x_1 - 3x_2 = 0$$

eq'n are same so $x_1 - x_2 = 0$

$$\frac{x_1}{1} = \frac{x_2}{1}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find principal direction :-

If $\lambda = 8$,
then

$$\cos^{-1}\left[\frac{1}{\sqrt{1^2+1^2}}\right] = \cos^{-1}\left[\frac{1}{\sqrt{2}}\right] = 45^\circ$$

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If $\lambda = 2$
then

$$\cos^{-1}\left[\frac{-1}{\sqrt{(-1)^2+1^2}}\right] = \cos^{-1}\left[\frac{-1}{\sqrt{2}}\right] = 135^\circ$$

$$\cos^{-1}\left[\frac{1}{\sqrt{1^2+(-1)^2}}\right] = \cos^{-1}\left[\frac{1}{\sqrt{2}}\right] = 45^\circ$$

These vectors make 45° & 135° angles with the positive x_1 - direction.

using polar codinates

Let

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$x = 8\cos\theta$ $\quad$ $y = 2\sin\theta$

$\cos\theta = \frac{x}{8}$ $\quad$ $\sin\theta = \frac{y}{2}$

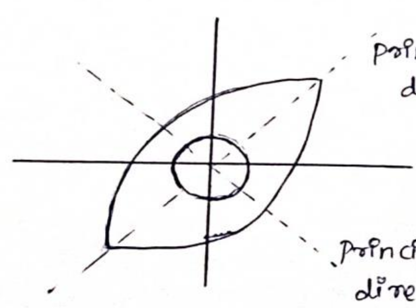
$$\cos^2\theta + \sin^2\theta = 1$$

$$\left(\frac{x}{8}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$\frac{x^2}{8^2} + \frac{y^2}{2^2} = 1$ which is the eq'n of ellipse

$x = 64$

$y = 4$



undeformed & deformed membrane.