Reduce the quadratic form:-

$Q = 3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$ form by orthogonal transformation. Find rank, index, signature and nature.

Step 1:-

Matrix forms

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

Step:- 2 The characteristic equation

$$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$$

Here, D_1 = sum of the main diagonal elements.

$$= 3+2+3 = 8$$

D_2 = sum of the minors of the main diagonal elements.

$$= (6-1) + (9-0) + (6-1)$$
$$= 5+9+5 = 19$$

$D_3 = |A|$

$$= 3(6-1) - (-1)(-3-0)$$
$$= 3(5) + 1(-3) = 15-3 = 12$$

$$\therefore \lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$$

Eigen values

$$\begin{array}{c|cccc} 1 & 1 & -8 & 19 & -12 \\ & 0 & 1 & -7 & 12 \\ \hline & 1 & -7 & 12 & \boxed{0} \end{array}$$

$\lambda' = 1$

$$\lambda^2 - 7\lambda + 12 = 0$$
$$(\lambda-4)(\lambda-3)$$
$$\lambda = 4, 3$$

12, -4, -3

$\therefore$ Eigen values are 1, 3, 4

Eigen vectors :-

$$(A-\lambda I)X = 0$$

$$\begin{pmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case 1 when $\lambda = 1$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -1 & 0 & 2 & -1 \\ 1 & -1 & -1 & 1 \end{array}$$

$$\frac{x_1}{(1-0)} = \frac{x_2}{(0+2)} = \frac{x_3}{(2-1)}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Case-2 when $\lambda = 3$

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -1 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 \end{array}$$

$$\frac{x_1}{(1+0)} = \frac{x_2}{0} = \frac{x_3}{(0-1)}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

<u>case:- 3</u> when $\lambda = 4$

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -2 \end{array}$$

$$\frac{x_1}{(1+0)} = \frac{x_2}{(0-1)} = \frac{x_3}{(2-1)}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

<u>Step:- 3</u>

$$X_1^T \cdot X_2 = (1\ 2\ 1)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1 + 0 - 1 = 0$$

$$X_2^T \cdot X_3 = (1\ 0\ -1)\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$X_3^T \cdot X_1 = (1\ -1\ 1)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 - 2 + 1 = 0$$

$\therefore$ Eigen vectors are pairwise orthogonal.

<u>Step :- 4</u> Normalized eigen vectors

$x_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\quad l(x) = \sqrt{(1)^2+(2)^2+(1)^2} = \sqrt{6}$

$$x_1 = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

$x_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\quad l(x) = \sqrt{(1)^2+(0)^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$

$$x_2' = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

$x_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\quad l(x) = \sqrt{(1)^2+(-1)^2+(1)^2} = \sqrt{3}$

$$x_3 = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

Step 5 :-

Normalized model matrix

$$N = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & \frac{1}{\sqrt{3}} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$$

Step 6 :-

N should be orthogonal

i.e, $NN^T = N^T N = I$

Now,

$$NN^T = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

N is orthogonal

Step :- 7 :- $D = N^T A N$

$$= \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Step 8 :- canonical form (CF) $= Y^T D Y$

$$CF = (y_1 \; y_2 \; y_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$y_1^2 + 3y_2^2 + 4y_3^2$$

→ Rank = 3

→ Index = 3

→ signature → 3

→ Nature ⇒ positive definite

2) Reduce the quadratic form
$Q = x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_1x_3$
into canonical form using orthogonal