Type 1 :

Find the envelope of $y = mx + \frac{a}{m}$, m being the parameter.

Soln.

$$y = mx + \frac{a}{m}$$

$$my = m^2x + a$$

$m^2x - my + a = 0$ which is the quadratic eqn. with parameter m.

$Ax^2 + Bx + C = 0 \Rightarrow B^2 - 4AC = 0$

Now, $A = x$, $B = -y$, $C = a$

$\therefore y^2 - 4ax = 0$ which is the envelope.

Find the envelope of $y = mx + \sqrt{a^2m^2 - b^2}$, m is the parameter.

Soln.

Given. $y = mx + \sqrt{a^2m^2 - b^2}$

$$y - mx = \sqrt{a^2m^2 - b^2}$$

Squaring on both sides,

$$(y - mx)^2 = a^2m^2 - b^2$$

$$y^2 + m^2x^2 - 2mxy = a^2m^2 - b^2$$

$$y^2 + m^2x^2 - 2mxy - a^2m^2 + b^2 = 0$$

$$m^2(x^2 - a^2) - 2xym + (y^2 + b^2) = 0$$

Here $A = x^2 - a^2$, $B = -2xy$, $C = y^2 + b^2$

Now $B^2 - 4AC = 0$

$$4x^2y^2 - 4(x^2 - a^2)(y^2 + b^2) = 0$$

$$4x^2y^2 - 4[x^2y^2 + x^2b^2 - a^2y^2 - a^2b^2] = 0$$

$\div 4$ $\quad x^2y^2 - x^2y^2 - x^2b^2 + a^2y^2 + a^2b^2 = 0$

$\div -(a^2b^2)$ $\quad -x^2b^2 + a^2y^2 = -a^2b^2$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$ which is the envelope.

Type 2:

1. Find the envelope of $\frac{x}{a}+\frac{y}{b}=1$ subject to $a+b=c$, where c is a constant.

Soln.

Given. $\frac{x}{a}+\frac{y}{b}=1 \rightarrow (1)$

$a+b=c \rightarrow (2)$

Differentiate (1) w.r.to 'a'

$$x\left[-\frac{1}{a^2}\right]+y\left[-\frac{1}{b^2}\right]\frac{db}{da}=0$$

$$-\frac{x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0$$

$$-\frac{y}{b^2}\frac{db}{da}=\frac{x}{a^2}$$

$$\frac{db}{da}=-\frac{xb^2}{ya^2} \rightarrow (3)$$

Differentiate (2) w.r. to a

$$1+\frac{db}{da}=0$$

$$\frac{db}{da}=-1 \rightarrow (4)$$

From (3) and (4),

$$-\frac{xb^2}{ya^2}=-1$$

$$\frac{x}{a^2}=\frac{y}{b^2}$$

$$\frac{\frac{x}{a}}{a}=\frac{y/b}{b}=\frac{\frac{x}{a}+\frac{y}{b}}{a+b}=\frac{1}{c}$$

$\therefore \frac{x}{a^2}=\frac{1}{c}$ and $\frac{y}{b^2}=\frac{1}{c}$

$a^2 = xc$ | $b^2 = yc$

$a = (xc)^{1/2}$ | $b = [yc]^{1/2}$

Subs in (2),

$(xc)^{1/2} + (yc)^{1/2} = c$

$c^{1/2}[x^{1/2} + y^{1/2}] = c$

$x^{1/2} + y^{1/2} = c^{1/2}$

3J. Find the envelope of $\frac{x}{a} + \frac{y}{b} = 1$ subject to $a^n + b^n = c^n$, where c is constant

Soln.

Given. $\frac{x}{a} + \frac{y}{b} = 1 \rightarrow (1)$

$a^n + b^n = c^n \rightarrow (2)$

Differentiate (1) w.r. to 'a'

$x\left(-\frac{1}{a^2}\right) + y\left(-\frac{1}{b^2}\right)\frac{db}{da} = 0$

$-\frac{y}{b^2}\frac{db}{da} = \frac{x}{a^2}$

$\frac{db}{da} = -\frac{xb^2}{ya^2} \rightarrow (3)$

Differentiate (2) w.r. to 'a'

$na^{n-1} + nb^{n-1}\frac{db}{da} = 0$

$nb^{n-1}\frac{db}{da} = -na^{n-1}$

$\frac{db}{da} = -\frac{a^{n-1}}{b^{n-1}} \rightarrow (4)$

From (3) and (4), $-\frac{xb^2}{ya^2} = -\frac{a^{n-1}}{b^{n-1}}$

$\frac{x}{a^{n-1}a^2} = \frac{y}{b^{n-1}b^2}$

$$\frac{x}{a^{n+1}} = \frac{y}{b^{n+1}}$$

$$\text{i.e., } \frac{x}{a^n} = \frac{y/b}{b^n} = \frac{x/a + y/b}{a^n + b^n} = \frac{1}{c^n}$$

$$\Rightarrow \frac{x}{a^{n+1}} = \frac{1}{c^n} \qquad \frac{y}{b^{n+1}} = \frac{1}{c^n}$$

$$a^{n+1} = xc^n \qquad b^{n+1} = yc^n$$

$$a^{(n+1)\times\frac{n}{n+1}} = (xc^n)^{n/(n+1)} \qquad b^{(n+1)\times\frac{n}{(n+1)}} = (yc^n)^{n/(n+1)}$$

$$a^n = (xc^n)^{n/(n+1)} \qquad b^n = (yc^n)^{n/(n+1)}$$

Subst. in (2)

$$(xc^n)^{n/(n+1)} + (yc^n)^{n/(n+1)} = c^n$$

$$c^{n^2/(n+1)}\left[x^{n/(n+1)} + y^{n/(n+1)}\right] = c^n$$

$$x^{n/(n+1)} + y^{n/(n+1)} = c^{n - n^2/(n+1)}$$

$$\frac{n^2+n-n^2}{n+1}$$

$$x^{n/(n+1)} + y^{n/(n+1)} = c^{n/(n+1)}$$

HW

1. Find the envelope of $\frac{x}{a} + \frac{y}{b} = 1$ subject to $a^2 + b^2 = c^2$ where c is a constant.

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2. Find the envelope of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subject to $a+b = c$, c is a constant.

Soln.

Given. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow (1)$

$a + b = c \longrightarrow (2)$

Differentiate (1) w.r. to a

$$x^2\left[\frac{-2}{a^3}\right] + y^2\left[\frac{-2}{b^3}\right]\frac{db}{da} = 0$$

$\div 2$ $\quad -\frac{y^2}{b^3}\frac{db}{da} = \frac{x^2}{a^3}$

$$\frac{db}{da} = -\frac{x^2 b^3}{a^3 y^2} \rightarrow (3)$$

Differentiate (2) w.r. to a

$$1 + \frac{db}{da} = 0$$

$$\frac{db}{da} = -1 \rightarrow (4)$$

From (3) & (4),

$$\frac{-x^2 b^3}{a^3 y^2} = -1$$

$$\frac{x^2}{a^3} = \frac{y^2}{b^3}$$

$$\frac{\frac{x^2}{a^2}}{a} = \frac{y^2}{b^2}$$

i.e., $\frac{\frac{x^2}{a^2}}{a} = \frac{\frac{y^2}{b^2}}{b} = \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{a+b} = \frac{1}{c}$

$\Rightarrow \frac{x^2}{a^3} = \frac{1}{c}$ $\quad a^3 = cx^2$ $\quad a = (cx^2)^{1/3}$

$\frac{y^2}{b^3} = \frac{1}{c}$ $\quad b^3 = cy^2$ $\quad b = (cy^2)^{1/3}$

Subs. in (2),

$$(cx^2)^{1/3} + (cy^2)^{1/3} = c$$

$$c^{1/3} x^{2/3} + c^{1/3} y^{2/3} = c$$

$$c^{1/3}\left[x^{2/3} + y^{2/3}\right] = c$$

$$x^{2/3} + y^{2/3} = c^1 c^{-1/3}$$

$$x^{2/3} + y^{2/3} = c^{2/3}$$

4). Find the envelope of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ subject to $a^n + b^n = c^n$, c is constant.

Soln.

Given. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow (1)$

$a^n + b^n = c^n \longrightarrow (2)$

Diff. (1) w.r. to 'a',

$$x^2\left(\frac{-2}{a^3}\right) + y^2\left(-\frac{2}{b^3}\right)\frac{db}{da} = 0$$

$\div 2$ $$-\frac{y^2}{b^3}\frac{db}{da} = \frac{x^2}{a^3}$$

$$\frac{db}{da} = \frac{-x^2 b^3}{y^2 a^3} \longrightarrow (3)$$

Diff (2) w.r. to 'a',

$$na^{n-1} + nb^{n-1}\frac{db}{da} = 0$$

$\div n$ $$b^{n-1}\frac{db}{da} = -a^{n-1}$$

$$\frac{db}{da} = \frac{-a^{n-1}}{b^{n-1}} \longrightarrow (4)$$

From (3) & (4), $$\frac{-x^2 b^3}{y^2 a^3} = \frac{-a^{n-1}}{b^{n-1}}$$

$$\frac{x^2}{a^3 a^{n-1}} = \frac{y^2}{b^{n-1} b^3}$$

i.e., $$\frac{x^2}{a^{n+2}} = \frac{y^2}{b^{n+2}}$$

$$\frac{\frac{x^2}{a^2}}{a^n} = \frac{y^2/b^2}{b^n} = \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{a^n + b^n} = \frac{1}{c^n}$$

$$\Rightarrow \frac{x^2}{a^{n+2}} = \frac{1}{c^n} \quad \Big| \quad \frac{y^2}{b^{n+2}} = \frac{1}{c^n}$$

$$\therefore \frac{\frac{x}{a}}{1} = \frac{\frac{y}{b}}{1} = \frac{\frac{x}{a} + \frac{y}{b}}{1+1} = \frac{1}{2}$$

$$\therefore \frac{x}{a} = \frac{1}{2} \qquad \frac{y}{b} = \frac{1}{2}$$

$$a = 2x \qquad b = 2y$$

Subs. (a) & (b) in (2),

$$(2x)(2y) = c^2$$

$$4xy = c^2$$