Envelope :

The envelope of the family of curves is a curve which touches each membrane of a family.

Procedure to find the envelope of the family of curves:

Method 1:

If the family of curves is expressed as the quadratic form say $Aa^2 + Ba + C = 0$ where 'a' is the parameter. Then the envelope is given by

$B^2 - 4AC = 0$

Method 2:

* Differentiate given eqn. with respect to parameter.
* Eliminate the parameter from the given curve.

Problems based on envelope with one parameter

1\. Find the envelope of $y = mx + am^2$, m is parameter

Soln.

The given eqn. is $y = mx + am^2$ (m is parameter)

ie., $am^2 + xm - y = 0$, which is a quadratic eqn. in m.

The envelope is given by,

$$B^2 - 4AC = 0$$

Here $A = a, B = x, C = -y$

$$\therefore x^2 - 4a(-y) = 0$$

$$x^2 + 4ay = 0$$

2\. Find the envelope of $y = mx + \sqrt{a^2m^2 + b^2}$

Soln.

Given. $y = mx + \sqrt{a^2m^2+b^2} \Rightarrow y - mx = \sqrt{a^2m^2+b^2}$

Squaring on bothsides,

$$(y-mx)^2 = a^2m^2 + b^2$$

$\Rightarrow y^2 - 2mxy + m^2x^2 - a^2m^2 - b^2 = 0$

$(x^2 - a^2)m^2 - 2(xy)m + (y^2 - b^2) = 0$ which is a quadratic eqn. in m.

The envelope is gvn. by $B^2 - 4AC = 0$

Here $A = (x^2 - a^2)$, $B = -2xy$, $C = y^2 - b^2$

$4x^2y^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$

$\div 4$ $\quad x^2y^2 = (x^2 - a^2)(y^2 - b^2)$

3) Find the envelope of $\frac{x^2}{\alpha} + \frac{y^2}{1-\alpha} = 1$, where α is the parameter

Soln.

Gvn. $\frac{x^2}{\alpha} + \frac{y^2}{1-\alpha} = 1$

$(1-\alpha)x^2 + \alpha y^2 = \alpha(1-\alpha)$

$\Rightarrow (1-\alpha)x^2 + \alpha y^2 - \alpha + \alpha^2 = 0$

$\Rightarrow \alpha^2 + x^2 - \alpha x^2 + \alpha y^2 - \alpha = 0$

$\Rightarrow \alpha^2 + (-x^2 + y^2 - 1)\alpha + x^2 = 0$

Here $A = 1$, $B = (-x^2 + y^2 - 1)$, $C = x^2$

Now $B^2 - 4AC = 0$

$[-x^2 + y^2 - 1]^2 - 4(1)(x^2) = 0$

$x^4 + y^4 + 1 - 2x^2y^2 - 2y^2 + 2x^2 - 4x^2 = 0$

$x^4 + y^4 + 1 - 2x^2y^2 - 2y^2 - 2x^2 = 0$

4). Find the envelope of $x\cos\theta + y\sin\theta = a$, where θ is the parameter.

Soln.

Gvn. $x\cos\theta + y\sin\theta = a \longrightarrow (1)$

Differentiate partially w.r.to 'θ',

$-x\sin\theta + y\cos\theta = 0 \longrightarrow (2)$

squaring of (1) and (2) separately,

$[x\cos\theta + y\sin\theta]^2 + [-x\sin\theta + y\cos\theta]^2 = a^2 + 0$

$x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta + x^2\sin^2\theta + y^2\cos^2\theta$

$-2xy\sin\theta\cos\theta = a^2$

$x^2[\sin^2\theta + \cos^2\theta] + y^2[\sin^2\theta + \cos^2\theta] = a^2$

$$\Rightarrow \quad x^2(1) + y^2(1) = a^2$$

$$x^2 + y^2 = a^2$$

Hw

1. $x \sec\theta - y \tan\theta = a$
2. $\frac{x}{a} \cos t + \frac{y}{b} \sin t = 1$