Q. Find the eqn. of evolute of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$

Soln.

Parametric form

$$x = a\cos^3\theta \quad\Big|\quad y = a\sin^3\theta$$

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) \quad\Big|\quad \frac{dy}{d\theta} = 3a\sin^2\theta(\cos\theta)$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{-\sin\theta}{\cos\theta}$$

$$y_1 = -\tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left[\frac{dy}{dx}\right]\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}[-\tan\theta]\frac{1}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta}$$

$$y_2 = \frac{\sec^4\theta}{3a\sin\theta} = \frac{1}{3a}\csc\theta\sec^4\theta$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\cos^3\theta - \frac{(-\tan\theta)(1+\tan^2\theta)}{\frac{1}{3a}\csc\theta\sec^4\theta}$$

$$= a\cos^3\theta + 3a\frac{\sin\theta}{\cos\theta}\sin\theta\cos^4\theta\sec^2\theta$$

$$= a\cos^3\theta + 3a\sin^2\theta\cos^3\theta\frac{1}{\cos^2\theta}$$

$\bar{x} = a\cos^3\theta + 3a\sin^2\theta\cos\theta \rightarrow (1)$

$\bar{y} = y + \frac{1+y_1^2}{y_2}$

$= a\sin^3\theta + \frac{1+\tan^2\theta}{\frac{1}{3a}\csc\theta\sec^4\theta}$

$= a\sin^3\theta + 3a\frac{\sec^2\theta}{\csc\theta\sec^4\theta}$

$= a\sin^3\theta + 3a\frac{1}{\csc\theta\sec^2\theta}$

$\bar{y} = a\sin^3\theta + 3a\sin\theta\cos^2\theta \rightarrow (2)$

$(1)+(2)$

$\Rightarrow \bar{x}+\bar{y} = a\cos^3\theta + 3a\sin^2\theta\cos\theta + a\sin^3\theta + 3a\sin\theta\cos^2\theta$

$= a[\cos^3\theta + 3\sin^2\theta\cos\theta + 3\sin\theta\cos^2\theta + \sin^3\theta]$

$\bar{x}+\bar{y} = a[\cos\theta + \sin\theta]^3 \rightarrow (3)$

$(1)-(2)$

$\Rightarrow \bar{x}-\bar{y} = a\cos^3\theta + 3a\sin^2\theta\cos\theta - a\sin^3\theta - 3a\sin\theta\cos^2\theta$

$\bar{x}-\bar{y} = a[\cos\theta - \sin\theta]^3 \rightarrow (4)$

Taking power as $2/3$ on bothsides of (1) & (2)

$(\bar{x}+\bar{y})^{2/3} = a^{2/3}[\cos\theta+\sin\theta]^{3\times 2/3}$

$(\bar{x}+\bar{y})^{2/3} = a^{2/3}[\cos\theta+\sin\theta]^2 \rightarrow (5)$

and $(\bar{x}-\bar{y})^{2/3} = a^{2/3}[\cos\theta-\sin\theta]^2 \rightarrow (6)$

$(5)+(6)$

$\Rightarrow (\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = a^{2/3}[\cos\theta+\sin\theta]^2 + a^{2/3}[\cos\theta-\sin\theta]^2$

$= a^{2/3}[\cos^2\theta+\sin^2\theta+2\cos\theta\sin\theta+\cos^2\theta+\sin^2\theta - 2\sin\theta\cos\theta]$

$= a^{2/3}[1+1]$

$(\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = 2a^{2/3}$

Unit-3 continuation..

5]. Find the evolute of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Soln.

Take $x = a\sec\theta$ | $y = b\tan\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$ | $\frac{dy}{d\theta} = b\sec^2\theta$

Now $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$

$= b\sec^2\theta \times$

$\frac{dy}{dx} = \frac{b}{a}\frac{\sec\theta}{\tan\theta}$

$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left[\frac{dy}{dx}\right]\frac{d\theta}{dx}$

$= \frac{d}{d\theta}\left[\frac{b}{a}\frac{\sec\theta}{\tan\theta}\right]\frac{1}{a\sec\theta\tan\theta}$

$= \frac{b}{a}\left[\frac{\tan\theta\sec\theta\tan\theta - \sec\theta\sec^2\theta}{\tan^2\theta}\right]\frac{1}{a\sec\theta\tan\theta}$

$= \frac{b}{a}\left[\frac{\tan^2\theta\sec\theta - \sec^3\theta}{\tan^2\theta}\right]\frac{1}{a\sec\theta\tan\theta}$

$= \frac{b}{a}\left[\frac{\tan^2\theta - \sec^2\theta}{\tan^2\theta}\right]\sec\theta\,\frac{1}{a\sec\theta\tan\theta}$

$= \frac{b}{a}\,\frac{-1}{\tan^2\theta}\,\frac{1}{a\tan\theta}$

$\frac{d^2y}{dx^2} = -\frac{b}{a^2\tan^3\theta}$

$\bar{x} = x - \frac{y_1[1+y_1^2]}{y_2} = a\sec\theta - \frac{\frac{b}{a}\frac{\sec\theta}{\tan\theta}\left[1 + \frac{b^2\sec^2\theta}{a^2\tan^2\theta}\right]}{-\frac{b}{a^2\tan^3\theta}}$

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$$= a\sec\theta + \frac{b}{a}\frac{\sec\theta}{\tan\theta}\ \frac{a^2\tan^2\theta + b^2\sec^2\theta}{a^2\tan^2\theta} \times \frac{a^2\tan^3\theta}{b}$$

$$= a\sec\theta + \frac{a^2}{a}\sec\theta\tan^2\theta + \frac{b^2}{a}\sec\theta\sec^2\theta$$

$$= a\sec\theta + a\sec\theta\tan^2\theta + \frac{b^2}{a}\sec^3\theta$$

$$= a\sec\theta[1+\tan^2\theta] + \frac{b^2}{a}\sec^3\theta$$

$$= a\sec\theta[\sec^2\theta] + \frac{b^2}{a}\sec^3\theta = a\sec^3\theta + \frac{b^2}{a}\sec^3\theta$$

$$\bar{x} = \left[\frac{a^2+b^2}{a}\right]\sec^3\theta$$

$$\bar{y} = b\tan\theta + \frac{\left[1+\frac{b^2\sec^2\theta}{a^2\tan^2\theta}\right]}{-\frac{b}{a^2}\frac{1}{\tan^3\theta}}$$

$$= b\tan\theta - \frac{a^2}{b}\tan^3\theta\left[\frac{a^2\tan^2\theta + b^2\sec^2\theta}{a^2\tan^2\theta}\right]$$

$$= b\tan\theta - \frac{\tan\theta}{b}[a^2\tan^2\theta + b^2\sec^2\theta]$$

$$= b\tan\theta - \frac{a^2}{b}\tan^3\theta - b\sec^2\theta\tan\theta$$

$$= b\tan\theta[1-\sec^2\theta] - \frac{a^2}{b}\tan^3\theta$$

$$= -b\tan\theta\tan^2\theta - \frac{a^2}{b}\tan^3\theta = -b\tan^3\theta - \frac{a^2}{b}\tan^3\theta$$

$$= \left[b - \frac{a^2}{b}\right]\tan^3\theta$$

$$= \left[\frac{-b^2-a^2}{b}\right]\tan^3\theta$$

$$\bar{y} = -\frac{[a^2+b^2]}{b}\tan^3\theta$$

Now, $\bar{x} = \left[\frac{a^2+b^2}{a}\right]\sec^3\theta$

$$\Rightarrow \sec^3\theta = \frac{a\bar{x}}{a^2+b^2}$$

$$\Rightarrow \sec^2\theta = \left[\frac{a\bar{x}}{a^2+b^2}\right]^{2/3} \quad \to (1)$$

$\bar{y} = -\left[\frac{a^2+b^2}{b}\right]\tan^3\theta$

$$\Rightarrow \tan^3\theta = \frac{-b\bar{y}}{(a^2+b^2)}$$

$$\Rightarrow \tan^2\theta = \left[\frac{b\bar{y}}{a^2+b^2}\right]^{2/3} \quad \to (2)$$

Now (1) − (2),

$$\sec^2\theta - \tan^2\theta = \left[\frac{a\bar{x}}{a^2+b^2}\right]^{2/3} - \left[\frac{b\bar{y}}{a^2+b^2}\right]^{2/3}$$

$$\Rightarrow (a^2+b^2)^{2/3} = (a\bar{x})^{2/3} - (b\bar{y})^{2/3}$$

Replace $\bar{x}$ by x and $\bar{y}$ by y.

$$\Rightarrow (ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$$

Show that the evolute of the rectangular hyperbola $xy = c^2$ is $(x+y)^{2/3} - (x-y)^{2/3} = (4c)^{2/3}$

Soln.

Take $x = ct$ and $y = \frac{c}{t}$

$$\frac{dx}{dt} = c \qquad \frac{dy}{dt} = -\frac{c}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx} = -\frac{c}{t^2} \times \frac{1}{c}$$

$$= -\frac{1}{t^2}$$

$$\frac{d^2y}{dx^2} = \frac{2}{ct^3}$$

$$\bar{x} = ct + \frac{\frac{1}{t^2}\left[1 + \frac{1}{t^4}\right]}{\frac{2}{ct^3}} = ct + \frac{1}{t^2}\left[\frac{t^4+1}{t^4}\right]\frac{ct^3}{2}$$

$$= ct + \frac{c}{2t^3}[t^4+1]$$

$$= ct + \frac{ct}{2} + \frac{c}{2t^3}$$

$$\bar{x} = \frac{3ct}{2} + \frac{c}{2t^3}$$

$$\bar{y} = \frac{c}{t} + \frac{\left[1 + \frac{1}{t^4}\right]}{\frac{2}{ct^3}} = \frac{c}{t} + \frac{ct^3}{2}\left[\frac{t^4+1}{t^4}\right]$$

$$= \frac{c}{t} + \frac{c}{2t}[t^4+1] = \frac{c}{t} + \frac{ct^3}{2} + \frac{c}{2t}$$

$$\bar{y} = \frac{3c}{2t} + \frac{ct^3}{2}$$

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Now, $\bar{x}+\bar{y} = \frac{3ct}{2} + \frac{c}{2t^3} + \frac{3c}{2t} + \frac{ct^3}{2}$

$= \frac{c}{2}\left[3t + \frac{1}{t^3} + \frac{3}{t} + t^3\right]$

$\bar{x}+\bar{y} = \frac{c}{2}\left[t + \frac{1}{t}\right]^3 \longrightarrow (1)$

and $\bar{x}-\bar{y} = \frac{3ct}{2} + \frac{c}{2t^3} - \frac{3c}{2t} - \frac{ct^3}{2}$

$= \frac{c}{2}\left[3t + \frac{1}{t^3} - \frac{3}{t} - t^3\right]$

$\bar{x}-\bar{y} = -\frac{c}{2}\left[t - \frac{1}{t}\right]^3 \longrightarrow (2)$

Taking power as $2/3$ on both sides of (1) & (2)

$(\bar{x}+\bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left[t + 1/t\right]^2$

$(\bar{x}-\bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left[t - 1/t\right]^2$

Now, $(\bar{x}+\bar{y})^{2/3} - (\bar{x}-\bar{y})^{2/3}$

$= \left(\frac{c}{2}\right)^{2/3}\left[\left(t+\frac{1}{t}\right)^2 - \left(t - 1/t\right)^2\right]$

$= \left(\frac{c}{2}\right)^{2/3}(4)$

Replace $\bar{x}$ by x & $\bar{y}$ by y,

$(x+y)^{2/3} - (x-y)^{2/3} = 4\left(\frac{c}{2}\right)^{2/3} = 2^2\, 2^{-2/3}(c)^{2/3}$

$= 2^{4/3}\, c^{2/3}$

$= (4c)^{2/3} \quad (\because 2^{4/3} = 4^{2/3})$

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5]. Show that the evolute of cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is $x = a(t - \sin t)$; $y - 2a = a(1 + \cos t)$

Soln.

Given $x = a(t + \sin t)$ | $y = a(1 - \cos t)$

$\frac{dx}{dt} = a(1 + \cos t)$ | $\frac{dy}{dt} = a \sin t$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= a \sin t \times \frac{1}{a(1 + \cos t)}$$

$$= \frac{\sin t}{1 + \cos t}$$

$\because \sin 2\theta = 2 \sin\theta \cos\theta$

$\frac{1 + \cos 2\theta}{2} = \cos^2\theta$

$$= \frac{2 \sin t/2 \cos t/2}{2 \cos^2 t/2}$$

$$= \tan t/2$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left[\tan \frac{t}{2}\right] \frac{1}{a(1 + \cos t)}$$

$$= \sec^2 \frac{t}{2}\left(\frac{1}{2}\right)\frac{1}{a(1 + \cos t)}$$

$$= \frac{\sec^2 t/2}{2a[2\cos^2 t/2]}$$

$$= \frac{\sec^4 t/2}{4a}$$

$$\bar{x} = x - \frac{y_1(1 + y_1^2)}{y_2} = a(t + \sin t) - \frac{\tan t/2 \left[1 + \tan^2 t/2\right]}{\frac{\sec^4 t/2}{4a}}$$

$$= a(t + \sin t) - \frac{4a \tan\frac{t}{2}\left[\sec^2 t/2\right]}{\sec^4 t/2}$$

$= a(t+\sin t) - 4a \sin t/2 \cos t/2$

$= at + a\sin t - 2a[2\sin t/2 \cos t/2]$

$= at + a\sin t - 2a \sin 2 t/2$

$= at + a\sin t - 2a \sin t$

$= at - a\sin t$

$\bar{x} = a(t - \sin t) \rightarrow (1)$

$\bar{y} = y + \frac{1+y_1^2}{y_2}$

$= a(1-\cos t) + \frac{1+\tan^2 t/2}{\frac{\sec^4 t/2}{4a}}$

$= a(1-\cos t) + \frac{4a \sec^2 t/2}{\sec^4 t/2}$

$= a(1-\cos t) + 4a\cos^2 t/2$

$= a - a\cos t + 2a[2\cos^2 t/2]$

$= a - a\cos t + 2a[1+\cos t]$ $\quad \because \frac{1+\cos 2\theta}{2} = \cos^2\theta$

$= a - a\cos t + 2a + 2a\cos t$

$\bar{y} - 2a = a + a\cos t \rightarrow (2)$

Replace $\bar{x}$ by x & $\bar{y}$ by y in (1) & (2),

$x = a(t - \sin t)$ &

$(y-2a) = a + a\cos t = a(1+\cos t)$

Hw

1. Find the eqn. of evolute $x^{2/3} + y^{2/3} = a^{2/3}$
2. Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$; $y = a(1-\cos\theta)$ is another cycloid $x = a(\theta + \sin\theta)$; $y = -a(1-\cos\theta)$