Evolutes:

The locus of the centre of curvature of the gvn. curve is called the evolute of the curve.

Method of finding Evolute:

1. Write the parametric form of the gvn. eqn.
   $x = f(t)$; $y = g(t)$
2. Find the centre of curvature $C(\bar{x}, \bar{y})$
3. Eliminate the parameter t & write the eqn. interms of $\bar{x}$ & $\bar{y}$ only.
4. Replace $\bar{x}$ by x & $\bar{y}$ by y in the obtained eqn., the evolute of the gvn. curve.

1. Find the evolute of the parabola $y^2 = 4ax$

Soln.

Take $x = at^2$ and $y = 2at$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

Now $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 2a \cdot \frac{1}{2at}$

$$= \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right]\frac{dt}{dx} = \frac{d}{dt}\left[\frac{1}{t}\right] \cdot \frac{1}{2at}$$

$$= -\frac{1}{t^2} \times \frac{1}{2at}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{2at^3}$$

$$\bar{x} = x - \frac{y_1[1+y_1^2]}{y_2} = x - \frac{\frac{1}{t}\left[1+\frac{1}{t^2}\right]}{\frac{-1}{2at^3}}$$

$$= x + \frac{1}{t}\left[\frac{t^2+1}{t^2}\right] \times 2at^3$$

$$\bar{x} = x + 2a(t^2+1)$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = y + \frac{1+\frac{1}{t^2}}{\frac{-1}{2at^3}}$$

$$= y - \frac{(t^2+1)}{t^2} \times 2at^3$$

$$= y - 2at[t^2+1]$$

At $(at^2, 2at)$,

$$\bar{x} = at^2 + 2a(t^2+1) = at^2 + 2at^2 + 2a$$

$$\bar{x} = 3at^2 + 2a$$

and $\bar{y} = 2at - 2at[t^2+1]$

$= 2at - 2at^3 - 2at$

$\bar{y} = -2at^3$

Here $\bar{x} = 3at^2 + 2a \Rightarrow \bar{x} - 2a = 3at^2$

Taking cube on bothsides, $[\bar{x} - 2a]^3 = (3at^2)^3$

$$\frac{(\bar{x}-2a)^3}{27a^3} = t^6 \longrightarrow (1)$$

and $\bar{y} = -2at^3$

Taking square on bothsides,

$$\bar{y}^2 = 4a^2t^6 \Rightarrow \frac{\bar{y}^2}{4a^2} = t^6 \longrightarrow (2)$$

From (1) and (2),

$$\frac{(\bar{x}-2a)^3}{27a^3} = \frac{\bar{y}^2}{4a^2}$$

$$4(\bar{x}-2a)^3 = 27a\bar{y}^2$$

Replace $\bar{x}$ by x & $\bar{y}$ by y, we get $4(x-2a)^3 = 27ay^2$

$\therefore$ The evolute is $4(x-2a)^3 = 27ay^2$.

2]. Find the evolute of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Soln.

Take $x = a\cos\theta$ | $y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta$ | $\frac{dy}{d\theta} = b\cos\theta$

Now, $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$

$= b\cos\theta \cdot \frac{1}{-a\sin\theta}$

$$\frac{dy}{dx} = -\frac{b}{a}\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left[\frac{dy}{dx}\right]\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\left[-\frac{b}{a}\cot\theta\right]\frac{1}{-a\sin\theta}$$

$$= \frac{-b}{a^2}\csc^2\theta \ \csc\theta$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a^2}\csc^3\theta$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = \cancel{a\cos\theta} \quad -\frac{b}{a}\cot\theta$$

$$= a\cos\theta + \frac{\frac{b}{a}\cot\theta\left[1+\frac{b^2}{a^2}\cot^2\theta\right]}{-\frac{b}{a^2}\csc^3\theta}$$

$$= a\cos\theta - \frac{a[a^2+b^2\cot^2\theta]}{a^2}\ \frac{\cos\theta}{\sin\theta}\ \sin^3\theta$$

$$= a\cos\theta - \frac{1}{a}\cos\theta\sin^2\theta\left[a^2+b^2\frac{\cos^2\theta}{\sin^2\theta}\right]$$

$$= a\cos\theta - \frac{1}{a}\cos\theta\sin^2\theta\frac{[a^2\sin^2\theta+b^2\cos^2\theta]}{\sin^2\theta}$$

$$= a\cos\theta - \frac{1}{a}\cos\theta\,[a^2\sin^2\theta+b^2\cos^2\theta]$$

$$= a\cos\theta - a\cos\theta\sin^2\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta[1-\sin^2\theta] - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta\cos^2\theta - \frac{b^2}{a}\cos^3\theta = a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$\bar{x} = \left[\frac{a^2-b^2}{a}\right]\cos^3\theta$$

$$\bar{y} = y + \frac{[1+y_1^2]}{y_2}$$

$$= b\sin\theta + \frac{\left[1+\frac{b^2}{a^2}\cot^2\theta\right]}{-\frac{b}{a^2}\csc^3\theta}$$

$$= b\sin\theta - \frac{a^2 + \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}}{\frac{b}{a^2}\csc^3\theta}$$

$$= b\sin\theta - \frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta} \times \frac{a^2}{b}\sin^3\theta$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\cos^2\theta\sin\theta$$

$$= b\sin\theta[1-\cos^2\theta] - \frac{a^2}{b}\sin^3\theta$$

$$= b\sin\theta\sin^2\theta - \frac{a^2}{b}\sin^3\theta = b\sin^3\theta - \frac{a^2}{b}\sin^3\theta$$

$$= \left[b - \frac{a^2}{b}\right]\sin^3\theta$$

$$\bar{y} = \left[\frac{b^2-a^2}{b}\right]\sin^3\theta$$

Now $\bar{x} = \left[\frac{a^2-b^2}{a}\right]\cos^3\theta$

$\Rightarrow \cos^3\theta = \frac{a\bar{x}}{a^2-b^2}$

Taking power as 2/3 on bothsides

$\cos^2\theta = \left[\frac{a\bar{x}}{a^2-b^2}\right]^{2/3}$ → (1)

$\bar{y} = \left[\frac{b^2-a^2}{b}\right]\sin^3\theta$

$\Rightarrow \sin^3\theta = \frac{b\bar{y}}{b^2-a^2}$

$\Rightarrow \sin^2\theta = \left[\frac{b\bar{y}}{b^2-a^2}\right]^{2/3}$ → (2)

$(1)+(2) \Rightarrow \cos^2\theta + \sin^2\theta = \left[\frac{a\bar{x}}{a^2-b^2}\right]^{2/3} + \left[\frac{b\bar{y}}{b^2-a^2}\right]^{2/3}$

$$\frac{(a\bar{x})^{2/3} + (b\bar{y})^{2/3}}{(a^2-b^2)^{2/3}} = 1 \Rightarrow (a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}$$

Replace $\bar{x}$ by x & $\bar{y}$ by y,

$$\Rightarrow (ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$$