TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

TWO MARKS Q & A

UNIT-I FOURIER SERIES

UNIT-II FOURIER TRANSFORM

UNIT-III PARTIAL DIFFERENTIAL EQUATIONS

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UNIT –I

FOURIER SERIES

1)Explain Dirichlet's conditions.

Ans:

Any function f(x) can be developed as a Fourier series $\frac{a0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$ where a_0 , a_n ,b_n are constants ,provided

(i) f(x) is periodic , single-valued and finite .

(ii) f(x) has a finite number of discontinuities in any one period .

(iii) f(x) has at most a finite number of maxima and minima.

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2). State whether y= tan x can be expanded as a Fourier series . If so how ? If not why ?

Soln:

tan x cannot be expanded as a Fourier series .Since tan x not satisfies Dirichlet's conditions.(tan x has infinite number of infinite discontinuous).

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3) Find the sum of the Fourier series for

$$f\{x\}=\begin{cases} x, 0 < x < 1 \\ 2, 1 < x < 2 \end{cases} \quad \text{at } x=1 .$$

Soln:

X=1 is a point of discontinuity.

$f(1-) =\lim_{h\to 0} f(1-h)$

$=\lim_{h\to 0} 1-h$

$= 1$

$f(1+)=\lim_{h\to 0} f(1+h)$

$= \lim_{h \to 0} 2$

$=2$

Sum $= \frac{f(1-)+f(1+)}{2}$

$= \frac{1+2}{2} \quad = \frac{3}{2}.$

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4) If the Fourier series for the function $f(x) = \begin{cases} 0, 0 < x < \pi \\ \sin x, \pi < x < 2\pi \end{cases}$ is

$f(x) = \frac{-1}{\pi} + \frac{2}{\pi}\left[\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \cdots\right] + \frac{1}{2}\sin x$ .Deduce that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots, \infty = \frac{\pi-2}{4}$.

Soln:

Put $x = \frac{\pi}{2}$ is a point of continuity.

$$\therefore 0 = \frac{-1}{\pi} + \frac{2}{\pi}\left[\frac{-1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \cdots\right] + \frac{1}{2}$$

$$= \frac{-1}{\pi} - \frac{2}{\pi}\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \cdots\right] + \frac{1}{2}$$

$$\frac{1}{\pi} - \frac{1}{2} = -\frac{2}{\pi}\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \cdots\right]$$

$$\frac{2-\pi}{2\pi} = -\frac{2}{\pi}\left[\frac{+1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \cdots\right]$$

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \cdots = -\frac{(2-\pi)}{2\pi} \times \frac{\pi}{2} = \frac{\pi-2}{4}.$$

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5) Write a_0, a_n in the expansion of $x + x^3$ as a Fourier series in $(-\pi, \pi)$.

Soln:

Let $f(x) = x + x^3$

$f(-x) = (-x) + (-x)^3$

$= -x - x^3$

$= -(x+ x^3)$

$= -f (x)$

$\therefore$ f(x) is an odd function.

Hence $a_0 = 0$ and $a_n = 0$

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6) What is the constant term a_0 and the coefficient of cos nx , a_n in the Fourier expansion of $f(x) = x - x^3$ in $(-\pi, \pi)$

Soln:

$f(x) = x-x^3$

$f(-x) = (-x) -(-x)^3$

$=-x+x^3$

$=-(x-x^3)$

$=-f(x)$

$\therefore$ f(x) is an odd function.

Hence in the fourier series $a_0 = 0$ and $a_n=0$.

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7) Find the constant term in the Fourier series corresponding to $f(x)=\cos^2 x$ expressed in the interval $(-\pi, \pi)$.

Soln:

Given $f(x)=\cos^2 x = \frac{1+\cos 2x}{2}$

W.K.T $f(x)=\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

To find $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x \, dx$

$= \frac{2}{\pi}\int_{0}^{\pi} \cos^2 x \, dx$

$= \frac{2}{\pi}\int_{0}^{\pi} \frac{1+\cos 2x}{2} dx$

$= \frac{1}{\pi}\int_{0}^{\pi} (1+\cos 2x) \, dx$

$=\frac{1}{\pi}\left[x+\frac{sin2x}{2}\right]_0^{\pi}$

$=\frac{1}{\pi}[(\pi+0)-(0+0)]$

=1

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8) In the Fourier expansion of $f(x)=\begin{cases}1+\frac{2x}{\pi}, -\pi<x<0\\ 1-\frac{2x}{\pi}, 0<x<\pi\end{cases}$ in$(-\pi,\pi)$ find the value of b_n, the coefficient of sinnx.

Soln:

Given $f(x)=\begin{cases}1+\frac{2x}{\pi}, -\pi<x<0\\ 1-\frac{2x}{\pi}, 0<x<\pi\end{cases}$

$=\begin{cases}\varphi_1(x), -\pi<x<0\\ \varphi_2(x), 0<x<\pi\end{cases}$

Here $\varphi_1(x) = 1+\frac{2X}{\pi}$,$\varphi_2(X) = 1-\frac{2x}{\pi}$

$\varphi_1(-x) = 1+\frac{2(-x)}{\pi}$

$= 1-\frac{2x}{\pi} = \varphi_2(x)$

∴ Given function is an even function.

Hence the value of b_n =0

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9) If $f(x)=x^2+x$ is expressed as a Fourier series in the interval (-2,2) to which value this series converges at x=2.

Soln:

X=2 is a point of discontinuity in the extremum .

∴ $f(x)_{at\ x=2} = \frac{f(-2)+f(2)}{2}$

$=\frac{[(-2)^2+(-2)]+[2^2+2]}{2}$

$$=\frac{[4-2]+[4+2]}{2} = \frac{2+6}{2} = 4$$

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10)Find b_n in the expansion of x^2 as a Fourier series in $(-\pi, \pi)$.

Soln:

Given $f(x)=x^2$ is an even function in the interval $(-\pi, \pi)$.

$\therefore b_n=0$.

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11) If f(x) is an odd function defined in (-l,l), what are the values of a_0 and a_n?

Soln:

Given f(x) is an odd function in the interval (-l,l).

$\therefore a_0=0, \ a_n=0$.

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12) Find the Fourier constants b_n for xsinx in $(-\pi, \pi)$

Soln:

Given f(x)= xsinx in $(-\pi, \pi)$

f(-x)= (-x) sin(-x)

= (-x)(-sinx)

= xsinx

= f(x)

$\therefore$ f(x) is an even function.

Hence $b_n=0$.

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13) If $f(x) = \begin{cases} \cos x, & if\ 0 < x < \pi \\ 50, & if\ \pi < x < 2\pi \end{cases}$

and $f(x) = f(x+2\pi)$ for all x, find the sum of the Fourier series of f(x) at $x=\pi$.

Soln:

Given $f(x) = \begin{cases} \cos x, & if\ 0 < x < \pi \\ 50, & if\ \pi < x < 2\pi \end{cases}$

To find f(x) at $x=\pi$.

$x=\pi.$ *is a* discontinuous point in the middle.

$\therefore f(\pi) = \frac{f(\pi-)+f(\pi+)}{2}$ -----------------(1)

$f(\pi-) = \lim_{h\to 0} f(\pi - h) = \lim_{h\to 0} cos(\pi - h) = -1$

$f(\pi+) = \lim_{h\to 0} f(\pi + h) = \lim_{h\to 0} 50 = 50.$

$\therefore (1) \Rightarrow f(\pi) = \frac{-1+50}{2} = \frac{49}{2}.$

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14) Determine the value of a_n in the Fourier series expansion of $f(x) = x^3$ in $-\pi<x<\pi$.

Soln

Let $f(x) = x^3$

$f(-x) = (-x)^3$

$= -x^3$

$=-f(x).$

$\therefore$ f(x) is an odd function.

Hence $a_0= 0$ and $a_n=0$.

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15. If $f(x) = 2x$ in the interval (0,4), then find the value of a_2 in the Fourier series expansion.

Soln:

Here $2l = 4 \Rightarrow l = 2$.

W.K.T $a_n = \frac{1}{l}\int_0^{2l} f(x)\, cos\frac{n\pi x}{l}$ dx.

$a_n = \frac{1}{2}\int_0^{4} f(x)\, cos\frac{n\pi x}{2}$ dx.

$a_2 = \frac{1}{2}\int_0^{4} 2x\, cos\frac{2\pi x}{2}$ dx.

$= \frac{1}{2}\int_0^{4} 2x\cos \pi x \; dx.$

$= \int_0^{4} x\cos \pi x \; dx.$

$= \left[x\left(\frac{sin\pi x}{\pi}\right) - (1)\left(\frac{-\cos \pi x}{\pi^2}\right)\right]_0^4$

$= \left[x\, sin\frac{\pi x}{\pi} + cos\frac{\pi x}{\pi^2}\right]_0^4$

$= \left[0 + \frac{1}{\pi^2}\right]$ - $\left[0 + \frac{1}{\pi^2}\right]$

$= 0.$

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16) Find Half range sine series for f(x) = k in $0 < x < \pi$.

Soln:

The sine series of f(x) in $(0,\pi)$ is given by

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} k \sin nx \; dx$

$= \frac{2k}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$

$= \frac{-2k}{n\pi}[\cos nx]_0^{\pi}$

$= \frac{-2k}{n\pi}[\cos n\pi - \cos 0]$

$= \frac{-2k}{n\pi}[(-1)^n - 1]$

$= \frac{2k}{n\pi} \quad [1-(-1)^n]$

$= 0$ when n is even.

$= \frac{4k}{n\pi}$ when n is odd.

$\therefore f(x) = \sum_{n=odd}^{\infty} \frac{4k}{n\pi} \sin nx.$

$= \frac{4k}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n} \sin nx$

$= \frac{4k}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)}.$

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17) Write the formula for Fourier constants for f(x) in the interval $(-\pi, \pi)$.

Soln:

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx.$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sinnx dx.$

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18) Find the constant a_0 of the Fourier series for the function f(x) = k , $0<x<2\pi$.

Soln:

$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx.$

$= \frac{1}{\pi} \int_0^{2\pi} k dx.$

$= \frac{k}{\pi} [x]_0^{2\pi}.$

$= \frac{2\pi k}{\pi}$

$= 2k.$

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19) If $f(x) = |x|$ expanded as a Fourier series in $-\pi < x < \pi$.Find a_0.

soln:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|dx.$$

$$=\frac{2}{\pi}\int_{0}^{\pi} xdx. \quad [since\ |x| \text{ is an even function}]$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}.$$

$\Rightarrow a_0 = \pi$.

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20) In the Fourier series expansion of $f(x) = |sinx|$ in $(-\pi, \pi)$. What is the value of b_n.

Soln:

Since $f(x) = |sinx|$ is an even function.

$\therefore$ $b_n = 0$.

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21) To which value , the half range sine series corresponding to $f(x) = x^2$ expressed in the interval (0,2) converges at x=2.

Soln:

Given $f(x) = x^2$.

X=2 is a point of discontinuity and also it is a point.

$$f(x) = \frac{f(0)+f(2)}{2}$$

$$=\frac{0+4}{2}$$

$$= 2.$$

The half range sine series corresponding to $f(x) = x^2$ expressed in the interval (0,2)

converges at x=2 is 2.
∴At x=2 , the series converges to 0.

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22) State parseval's identity for the half-range cosine expansion of f(x) in (0,1).

Ans:

$$\frac{2}{1}\int_0^1 [f(x)]^2 dx = \frac{a_0{}^2}{2} + \sum_{n=1}^{\infty} a_n{}^2$$

Where $a_0 = 2\int_0^1 f(x)dx$; $a_n = = 2\int_0^1 f(x)\cos nx\, dx$.

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23) Find the root mean square value of the function f(x) = x in the interval (0,l).

Soln:

R.M.S = $\sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$ in the interval (a,b).

$= \sqrt{\frac{\int_0^l x^2 dx}{l-0}}$ Here a=0 , b=l.

$= \sqrt{\frac{1}{l}\left[\frac{x^3}{3}\right]_0^l}$

$= \sqrt{\frac{1}{l}\left(\frac{l^3}{3} - 0\right)} = \sqrt{\frac{l^2}{3}} = \frac{l}{\sqrt{3}}$.

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24) Define root mean square value of a function f(x) in a< x< b.

Soln:

Let f(x) be a function defined in an interval (a,b) then $\sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$ is called the root mean square value (or) effective value of f(x) and is denoted by $\overline{y}$.

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25) What do you mean by Harmonic Analysis?

Soln:

The process of finding the Fourier series for a function given by numerical value is known as Harmonic Analysis . In Harmonic Analysis the Fourier coefficients a_0 ,a_n and b_n of the function $y= f(x)$ in $(0,2\pi)$ are given by

$a_0 = 2$ [mean value of y in$(0,2\pi)$]

$a_n = 2$[mean value of y cos nx in $(0,2\pi)$]

$b_n = 2$[mean value of y sin nx in $(0,2\pi)$].

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UNIT-II - FOURIER TRANSFORMS

1.State Fourier Integral theorem.

Statement:

If f(x) is piecewise continuously differentiable & absolutely integrable in$(-\infty,\infty)$ then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{is(x-t)}\, dt\, ds.$

(or)

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t) \cos \lambda(t-x)\, dt\, d\lambda$$

This is known as Fourier integral theorem.

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2.Show that $f(x) =1 ,0 < x < \infty$ cannot be represented by a fourier integral.

Solution :

$$\int_{0}^{\infty} | f(x) |\, dx = \int_{0}^{\infty} 1.\, dx = \left[x\right]_{0}^{\infty} = \infty .$$

and this value tends to ∞ . as $x \to \infty$.

ie) $\int_0^{\infty} f(x)\,dx$ is not convergent.

Hence f(x) =1 cannot be represented by a Fourier integral

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3. Define Fourier transform pair. (or)

Define Fourier transform and its inverse transform.

Ans:

The complex Fourier transform of f(x) is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

Then the function f(x) is the inverse Fourier transform of F(s) is

Given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds.$$

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4. What is the Fourier cosine transform & inverse cosine transform of a function?

Solution:

The infinite Fourier cosine transform of f(x) is defined by

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\, \cos sx\, dx.$$

The inverse Fourier cosine transform $F_c[f(x)]$ is defined by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c[f(x)]\, \cos sx\, ds.$$

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5. Find the Fourier cosine transform of $f(x) = \begin{cases} \cos x & \text{if } ,0 < x < a \\ 0 & \text{if } x \geq a. \end{cases}$

Solution:

$$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a \cos x\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^a [\cos(s+1)x + \cos(s-1)x]\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)a}{(s+1)} + \frac{\sin(s-1)a}{(s-1)} - (0+0)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)a}{(s+1)} + \frac{\sin(s-1)a}{(s-1)}\right]$$

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6. Find Fourier cosine transform of $e^{-ax.}$

Solution:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx.$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]$$

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7. Find Fourier cosine transform of $e^{-x.}$

Solution:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx.$$

$F_c\,[e^{-x}] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\ \cos sx\ dx$

$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{1+a^2}\right]$

8. . Find Fourier sine transform of $e^{-3x.}$

Solution: $F_s[e^{-3x]} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-3x}\ \sin sx\ dx$

$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+3^2}\right]$

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9. Find Fourier sine transform of $3e^{-2x.}$

Solution:

Let $f(x) = 3e^{-2x}$

$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \sin sx\ dx$

$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} 3e^{-2x} \sin sx\ dx$

$= 3\ \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-2x} \sin sx\ dx$

$= 3\sqrt{\frac{2}{\pi}}\left[\frac{e^{-2x}}{4+s^2}(-2 sinsx - s\cos sx\right]_0^{\infty}$

$= 3\sqrt{\frac{2}{\pi}}\left[0-\frac{1}{4+s^2}(-s)\right]$

$=3\sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+4}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{3s}{s^2+4}\right]$

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10. Find Fourier sine transform of 1/x

Solution:

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \sin sx \, dx$$

$$F_s[1/x] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} 1/x \sin sx \, dx$$

Let sx = ∞ . $x \to 0 \Rightarrow \theta \to 0$

S dx = d θ $x \to \infty \Rightarrow \theta \to \infty$.

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{s}{\theta} \sin\theta \frac{d\theta}{s}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{\sin\theta}{\theta} d\theta$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2}\right]$$

$$= \sqrt{\frac{\pi}{2}}$$

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11. Define Find Fourier sine transform and its inversion formula.

Ans:

The infinite Fourier sine transform of f(x) is defined by

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \sin sx \, dx$$

The inverse Fourier sine transform of F_S[f(x)] is defined by

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s[f(x)] \sin sx \, ds$$

12.Find the Fourier sine transform of $f(x) = e^{-ax}$, a> 0 and hence deduce that

$$\int_0^\infty \frac{x \sin mx}{1+x^2} dx. = \frac{\pi}{2} e^{-\alpha} .$$

Solution:

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{1+s^2} \right]$$

By inversion formula

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[e^{-x}] \sin sx \, ds.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{s}{1+s^2} \right] \sin sx \, ds.$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{1+s^2} ds.$$

$$\frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{1+s^2} ds. = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} e^{-x}.$$

Changing x to & s to x we get

$$\int_0^\infty \frac{x \sin mx}{1+x^2} dx. = \frac{\pi}{2} e^{-m}.$$

13. If Fourier transform of f(x) = F(s) then what is Fourier transform of f(ax)
Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(x)\ e^{isx}\ dx.$$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(ax)\ e^{isx}\ dx$$

Put t = ax x → -∞ => t → - ∞ .
dt = a dx x → ∞ => t→ ∞ .

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\ e^{ist/a}\ dt/a$$

$$=1/a\ \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\ e^{is(t/a)}\ dt$$

$$=1/a\ \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(x)e^{is/ax}\ dx$$

= 1/a F[s/a]

$$F[f(ax)] = -\frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(ax)\ e^{isx}\ dx$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\ e^{ist/a}\ dt/a$$

$$= -1/a\ \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\ e^{is(t/a)}\ dt$$

$$= -1/a\ \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(x)e^{is/ax}\ dx$$

= -1/a F[s/a]

$$F[f(ax)] = \frac{1}{|a|} F[s/a].$$

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14.If Fourier transform of f(x) is F(s), P.T the Fourier transform of f(x) cosax is

1/2 [F(s-a) + F(s+a)].

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(x)\ e^{isx}\ dx$$

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(x)\cos ax\, e^{isx}\, dx$$

$$= 1/2\ \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(x)\ [e^{i(s+a)x} + e^{i(s-a)x}]\ dx$$

$$= 1/2\ [\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} [\ f(x)\, e^{i(s+a)x}\, dx\ +\ \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(x)\, e^{i(s-a)x}\ dx\ ]$$

$$= 1/2\ [F(s-a) + F(s+a)].$$

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15 . P.T $F_c[f(x)\ \cos ax] = 1/2\ [F_c(s+a) + F_c(s-a)]$ where F_c denotes the Fourier cosine transform of f(x).

Solution:

$$F_c[f(x)\ \cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \cos ax\ \cos sx\ dx.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \cos sx\ \cos ax\ dx$$

$$= 1/2\ [\ \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \cos(s+a)x\ dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \cos(s-a)x\ dx]$$

$$= 1/2\ [F_c(s+a) + F_c(s-a)]$$

■■■

16. If F(s) is the Fourier transform of f(x) then show that the Fourier transform of $e^{iax} f(x)$ is F(s+a).

solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(x)\, e^{isx}\, dx.$$

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{iax} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(x)\, e^{i(s+a)x}\, dx$$

$= F(s+a)$

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17. If F(s) is the complex Fourier transform of f(x) then find F[f(x-a)].

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(x)\, e^{isx}\, dx.$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(x-a)\, e^{isx}\, dx.$$

Put $t=x-a$ $\qquad$ $x \to -\infty \Rightarrow t \to -\infty$.

$dt = dx$ $\qquad$ $x \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\, e^{is(t+a)}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\, e^{ist}\, e^{isa}\, dt$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\, e^{ist}\, dt$$

$= e^{isa} F[f(t)]$

$= e^{isa} F(s)$

****

18.Given that $e^{-}x^2/2$ is self reciprocal under Fourier cosine transform , find

(i) Fourier sine transform of $x e^{-x^2/2}$ and

(i) Fourier cosine transform of $x^2 e^{-x^2/2}$

Solution:

$$F_c[e^{-x^2/2}] = e^{-s^2/2}$$

$$F_s[x e^{-x^2/2}] = \frac{-d}{ds} F_c[x e^{-x^2/2}]$$

$$= \frac{-d}{ds}[e^{-s^2/2}]$$

$$= -e^{-s^2/2}(-s)$$

$$= -s e^{-s^2/2}$$

$$F_s[x^2 e^{-x^2/2}] = \frac{d}{ds} F_s[x e^{-x^2/2}]$$

$$= \frac{d}{ds}[s e^{-s^2/2}]$$

$$= [s e^{-s^2/2}(-s) + e^{-s^2/2}]$$

$$= (1-s^2)\, e^{-s^2/2}$$

---

19. State the convolution theorem for Fourier cosine transform.
    Statement:

    If F(s) &G(s) are the Fourier transform of f(x) & g(x) respectively, Then the Fourier transform of the convolution of f(x) & g(x) is the product of their Fourier transform

    F[f(x) * g(x)] = F(s) G(s) = F[f(x)] G[g(x)]

****

20. State the Fourier transform of the derivatives of a function.

    Statement:

    The Fourier transform of $F'(x)$

The derivatives of F(x) is f(x), where f(s) is the Fourier transform of F(x)

$$F[F'(x)] = isf(s)$$

****

21.Find the Fourier sine transform of f(x) e^{-x}

Solution:

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \sin sx \, dx$$

$$F_s[e^{-x}] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \, [s/1+s^2]$$

22.Give a function which self reciprocal under Fourier sine & cosine transforms

Solution:

$=1/\sqrt{x}$

****

23.State the modulation theorem in Fourier transform

Statement:

If F(s) is the Fourier transform of f(x) , then

$$F[f(x) \cos ax] = 1/2 \, [F(s+a) + F(s-a)].$$

****

24.State the Parsevals identity on Fourier transform

Statement:

If F(s) is the Fourier transform of f(x), then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

****

25. Find $F_c\,[xe^{-ax}]$

Solution:

$$F_c\,[xe^{-ax}] = F_s\,[e^{-ax}]$$

$$= \left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{1+s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}}$$

****

UNIT:III

PARTIAL DIFFERENTIAL EQUATIONS

1)Explain how partial differential equation are formed.

Soln:

Partial differential equation can be obtained

i)by eliminating the arbitrary constants that occure in the functional relation between the dependent and independent variables. (OR)

ii)by eliminating arbitrary functions from a given relation between the dependent and independent variables.

^^^

2) Form the partial differential equation by eliminating the arbitrary constants a and b from Z=ax+by.

Soln:

Given Z=ax+by ---------(1)

Differentiating (1) partially w.r.to 'x' we get

$\frac{\partial z}{\partial x} = a$, ie) p = a

Differentiating (1) partially w. r.to 'y' we get

$\frac{\partial z}{\partial y} = b$, ie) q = b.

Substituting in (1) we get the required p.d.e z=px+qy.

^^^

3) Eliminate the arbitrary constants a and b from $z=ax+by+a^2+b^2$.

Soln:

Given $z=ax+by+a^2+b^2$ ---------(1)

Differentiating (1) partially w.r.to 'x' we get

$$\frac{\partial z}{\partial x} = a$$

ie) p = a -----------------(2)

differentiating (1) partially w.r.to 'y' we get

$$\frac{\partial z}{\partial y} = b,$$

ie) q = b -------------(3)

substituting in equation (1) we get the required p.d.e $z=px+qy+p^2+q^2$

^^^

4) Form a p.d.e by eliminating the arbitrary constants a and b from $Z=(x+a)^2+(y-b)^2$

Soln:

Given $Z= (x+a)^2+(y-b)^2$

$P = \frac{\partial z}{\partial x} = 2(x+a)$, ie) $x+a = \frac{p}{2}$

$q = \frac{\partial z}{\partial y} = 2(y-b)$, ie) $y-b = \frac{q}{2}$

$$\therefore (1) \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$z = \frac{p^2}{4} + \frac{q^2}{4}$$

$$4z = p^2+q^2$$

Which is the required p.d.e.

^^^

5) Form the p.d.e by eliminating the constants a and b from $z = ax^n+by^n$.

Soln:

Given: $z = ax^n+by^n$. ------------(1)

$$P = \frac{\partial z}{\partial x} = anx^{n-1}$$

$$\frac{p}{n} = ax^{n-1}$$

Multiply ' x' we get, $\frac{px}{n} = ax^n$ ---------(2)

$q = \frac{\partial z}{\partial y}$ = bny^{n-1}

$\frac{q}{n}$ = by^{n-1}

Multiply ' y' we get , $\frac{qy}{n}$ =by^n ----------(3)

Substitute (2) and (3) in (1) we get the required p.d.e $z = \frac{px}{n} + \frac{qy}{n}$

Zn=px+qy.

^^

6.Form the partial differential equation by eliminating a and b from z=a(x+y)+b.

Soln:

Given z = a(x+y)+b

$P = \frac{\partial z}{\partial x}$ =a -------------(1)

$q = \frac{\partial z}{\partial y}$ =a --------------(2)

From (1) and (2) we get the required p.d.e p = q.

^^

7) Find the p.d.e of all planes having equal intercepts on the X and Y axis.

Soln:

Intercept form of the plane equation is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Given : a=b. [Equal intercepts on the x and y axis]

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$..------------(1)

Here a and c are the two arbitrary constants .

Differentiating (1) p.w.r.to 'x' we get

$$\frac{1}{a}+0+\frac{1}{c}\frac{\partial z}{\partial x}=0$$

$$\frac{1}{a}+\frac{1}{c}p=0.$$

$$\frac{1}{a}=-\frac{1}{c}p$$.-----------------(2)

Dff(1) p.w.r.to. 'y' we get

$$0+\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial y}=0.$$

$$\frac{1}{a}+\frac{1}{c}q=0$$

$$\frac{1}{a}=-\frac{1}{c}q \text{ .--------------------(3)}$$

From (2) and (3) $\Rightarrow -\frac{1}{c}p=-\frac{1}{c}q$

p = q ,whih is the required p.d e.

^^^

8) Eliminate f from z = x+y+f(xy)

Soln:

Given z = x+y+f(xy) -----------(1)

Diff (1) p.w.r.to 'x'

P = $\frac{\partial z}{\partial x}$ =1+f '(xy)y

p-1 = yf ′(xy) ---------------(2)

diff (1) p.w.r.to 'y'

q = $\frac{\partial z}{\partial y}$ =1+f ′(xy)x

q-1 = xf ′(xy) -----------------(3)

$$\frac{(2)}{(3)}\Rightarrow\frac{p-1}{q-1}=\frac{y}{x}$$

Px-x = qy-y

Px-qy = x-y is the required p.d.e.

^^^

9) Eliminate the arbitrary function f from z= $f\left(\frac{y}{x}\right)$ and form a partial differential equation.

Soln:

Given z = $f\left(\frac{y}{x}\right)$ ----------------(1)

Differentiating (1) p.w.r.to 'x' we get

P = $\frac{\partial z}{\partial x}$ = $f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$ --------------(2)

Differentiating (1) p.w.r.to y we get

q = $\frac{\partial z}{\partial y}$ = $f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$-----------------(3)

$$\frac{(2)}{(3)}\Rightarrow\frac{p}{q}=\frac{-y}{x}$$

$\therefore$ px = -qy

ie) px+qy = 0 is the required p.d.e.

^^

10) Eliminate the arbitrary functions f and g from z = f(x+iy)+g(x-iy) to obtain a partial differential equation involving z,x,y.

Soln:

Given : z = f(x+iy)+g(x-iy) -------------------(1)

$P = \frac{\partial z}{\partial x} = f'(x+iy)+g'(x-iy)$ ----------------(2)

$q = \frac{\partial z}{\partial y} = i f'(x+iy)-ig'(x-iy)$ ----------------(3)

$r = \frac{\partial^2 z}{\partial x^2} = f''(x+iy)+g''(x-iy)$ ----------------(4)

$t = \frac{\partial^2 z}{\partial y^2} = -f''(x+iy)-g''(x-iy)$ ----------------(5)

r+t=0 is the required p.d.e.

^^

11) Find the general solution of $\frac{\partial^2 z}{\partial y^2} = 0$

Soln:

Given $\frac{\partial^2 z}{\partial y^2} = 0$

ie) $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = 0$

Integrating w.r.to 'y' on both sides

$\frac{\partial z}{\partial y} = a$ (constants)

ie) $\frac{\partial z}{\partial y} = f(x)$

Again integrating w.r.to 'y' on both sides.

z =f(x)y+b

ie) z=f(x)y+F(x)

(or) z=y f(x)+F(x) , where both f(x) and F(x) are arbitrary.

^^

12) Mention three types of solution of a p.d.e (or) Define general and complete integrals of a p.d.e.

soln:

1) A solution which contains as many arbitrary constants as there are independent variables is called a complete integral (or)complete solution.(number of arbitrary constants=number of independent variables)
2) A solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral (or) particular solution.
3) A solution of a p.d.e which contains the maximum possible number of arbitrary functions is called a general integral (or) general solution.

^^

13) Find the complete integral of p-q=o.

Soln:

Given p-q=0 -----------------(1)

This equation of the form F(p,q)=0---------(2)

Hence the trial soln is z=ax+by+c ---------(3)

To get the complete integral (solution) of (3).

We have to eliminate any one of the arbitrary constants. Since in a complete integral.

Number of arbitrary constants=number of independent variables

(3) $\Rightarrow$ z = ax+by+c

$P = \frac{\partial z}{\partial x} = a$

$q = \frac{\partial z}{\partial y} = b$

$\therefore$ (1) $\Rightarrow$ a-b=0

b = a

Hence the complete integral is z = ax+ay+c.

Hence number of arbitrary constants=number of independent variables.

^^

14) Obtain the complete solution of the equation $z = px+qy-2\sqrt{pq}$.

Soln:

Given :$z = px+qy-2\sqrt{pq}$

This is of the form z = px+qy+f(p,q) ,[clairaut's form]

Hence the complete integral is

$z = ax+by-2\sqrt{ab}$, where a and b are arbitrary constants.

^^

15) Find the complete integral of the partial differential equation (1-x)p+(2-y)q =3-z.

Soln:

Given (1-x)p+(2-y)q = 3-z

p-px+2q-qy = 3-z

z = px+qy-p-2q+3

This equation is of the form z = px+qy+f(p,q) , [clairaut's type]

Hence the complete integral is z = ax+by-a-2b+3.

^^

16) Solve p =2qx.

Soln:

Given p =2qx, this equation is of the form f(x,p,q) = 0.

Let q=a

Then p=2ax

But dz = pdx+qdy

dz = 2ax.dx+ady

Integrating on both sides we get

$z=ax^2+ay+c$ ---------------------(1)

equation (1) is the complete integral of the given equation.

Differentiating partially w.r.to 'c' ,we get 1=0

Hence there is no singular integral .

General integral can be found out in the usual way.

^^

17)Find the complete integral of q = 2px.

Soln:

Given q = 2px

This eqn is of the form f(x,p,q)=0

Let q = a then $p = \frac{a}{2x}$

But $dz = \frac{a}{2x}dx+ady$

Integrating on both sides

$\int dz = \int \frac{a}{2x} dx+\int a\, dy$

$z = \frac{a}{2} \log x+ay+b.$

^^

18) Find the complete integral of pq = xy.

Soln:

Given pq = xy

Hence $\frac{p}{x} = \frac{y}{q}$

It is of the form $f(x,p)=\phi(y,q)$

Let $\frac{p}{x} = \frac{y}{q}$=a [a is an arbitrary constant]

P =ax and $q=\frac{y}{a}$

Hence dz = p dx+qdy

dz = axdx+ $\frac{y}{a}$ dy

Integrating on both sides.

z = a $\frac{x^2}{2}+\frac{y^2}{2a}+c$

2az = $a^2x^2+y^2+b$ is the required complete integral.

^^

19) Find the complete integral of $\sqrt{p}+\sqrt{q}$ =2x

Soln:

Given $\sqrt{p}+\sqrt{q}$ =2x

The given equation can be written as

$\sqrt{p}$ -2x= $-\sqrt{q}$

This is of the form f(x,p) = ϕ(y,q)

Let $\sqrt{p}$ -2x= - $\sqrt{q}$ = a (say)

$\sqrt{p}$ =a+2x, $\sqrt{q}$ = -a

P=$(a+2x)^2$, q=a^2

Now dz = p dx- q dy

= $(a+dx)^2$ dx+a^2 dy

z = $\frac{(a+2x)^3}{6}$ +a^2y+b is the complete integral.

^^^

20) Solve px+qy=z

Soln:

Given px+qy=z -----------(1)

This eqn is of the form $P_p = Q_q$ =R

When P = x, Q = y ,R = z

The subsidiary equations are $\frac{dx}{p}=\frac{dy}{Q}=\frac{dz}{R}$

ie) $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$

Take $\frac{dx}{x}=\frac{dy}{y}$, Take $\frac{dx}{x}=\frac{dz}{z}$

$\int\frac{dx}{x}=\int\frac{dy}{y}$, $\int\frac{dx}{x}=\int\frac{dz}{z}$

$\log x = \log y + \log c_1$, $\log x = \log z + \log c_2$

$\log x = \log(yc_1)$, $\log x = \log(zc_2)$

$x = yc_1$, $x = zc_2$

$\frac{x}{y} = c_1$, $\frac{x}{z} = c_2$

ie) $u = \frac{x}{y}$, $v = \frac{x}{z}$

$\therefore$ The solution of the given p.d.e is $\phi\left(\frac{x}{y}, \frac{x}{z}\right) = 0$

^^

21) Solve $(D^2 - 2DD' + D'^2)z = 0$

Soln:

Given $(D^2 - 2DD' + D'^2)z = 0$

The auxiliary eqn is $m^2 - 2m + 1 = 0$

ie) $(m-1)^2 = 0$

$m = 1, 1$

The roots are equal.

$\therefore$ C.F $= \phi_1(y+x) + x\phi_2(y+x)$

Hence z = C.F alone

$z = \phi_1(y+x) + x\phi_2(y+x)$.

^^

22) Solve $(D^4 - D'^4)z = 0$

Soln:

Given $(D^4 - D'^4)z = 0$

The auxiliary equation is $m^4 - 1 = 0$

[Replace D by m and D′ by 1]

Solving $(m^2-1)(m^2+1) = 0$

$m^2 - 1 = 0$, $m^2 + 1 = 0$

$m^2 = 1$, $m^2 = -1$

$m = \pm 1$, $m = \pm\sqrt{-1} = \pm i$

ie) $m = 1, -1, i, -i$

The solution is $z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+ix) + \phi_4(y-ix)$.

^^

23) Find the P.I of $[D^2 + 4DD']y = e^x$

Soln:

$$\text{P.I} = \frac{1}{D^2 + 4DD'} e^x$$

$= \frac{1}{D^2+4DD'} e^{x+0y}$

$= e^{x}\left[\frac{1}{1+4(1)(0)}\right]$ Replace D by 1 and D' by 0

$= e^{x}$.

^^

24) Solve $[D^2-2DD'+D'^2]$ z=cos(x-3y).

Soln:

Given $[D^2-2DD'+D'^2]$ z=cos(x-3y).

The auxiliary equation is $m^2-2m+1=0$

$(m-1)^2 = 0$

m =1,1

C.F $=f_1(y+x)+xf_2(y+x)$.

$$\text{P.I} = \frac{1}{D^2-2DD'+D'^2}\cos(x-3y)$$

$$= \frac{\cos(x-3y)}{-1-2(3)-9}$$

$$= \frac{-1}{16}\cos(x-3y)$$

$\therefore$ The complete solution is $Z = f_1(y+x)+xf_2(y+x)-\frac{1}{16}\cos(x-3y)$.

^^

25) Solve (D+D′-2)z = 0

Soln:

Given (D+D′-2)z = 0

ie) [D-(-1)D′-2]z = 0

we know that working rule case (i) is

If (D-mD′-c)z = 0 then $z = e^{cx}$ f(y+mx) where f is arbitrary

Here m = -1, c =2

$\therefore$ $z = e^{2x}$ f[y+(-1)x] = e^{2x} f(y-x)

^^

UNIT-IV

APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS.

1. Write down all possible solutions of one dimensional wave equation.

Ans:

$y(x,t)=(c_1e^{px}+c_2e^{-px})\ (c_3e^{pat}+c_4e^{-pat})$

$y(x,t)=(\ c_5\cos px+c_6\sin px)\ (c_7\cos pat+c_8\sin pat)$

$y(x,t)=(c_9x+c_{10})(c_{11}t+c_{12})$.

≈≈≈

2.Classify the partial differential equation $4u_{xx}=u_{t}$

Ans:

Given $4u_{xx}-u_t=0$

$A=4,\ B=0,C=0$

$\Delta=\ B^2-4AC=(0)2-4(4)0$

$=0$

p.d.e is parabolic.

≈≈≈

3.Classify the partial differential equation $x^2u_{xx}+2xyu_{xy}+(1+y^2)u_{yy}-2u_x=0$

Ans:

$A=x^2,B=2xy,C=1+y^2$

$\Delta=B^{2-}-4AC$

$=-4x^2\ \ <0$

p.d.e. is elliptic.

≈≈≈

4.Classify the partial differential equation $u_{xx}=u_{yy}$

Ans:

$A=1,B=0,C=-1$

$\Delta=B^2-4AC$

$=0-4(1)(-1)$

$=4$

>0

p.d.e is hyperbolic.

≈≈≈

5.A rod 20cm long with insulated sides has its ends A and B kept at 30^oC and 90^oC respectively. Find the steady state temperature distribution of the rod.

Ans:

When steady state condition exists the heat flow equation is

$$U_{xx}=0$$

i.e., $U(x)=c_1x+c_2$…………………(1)

The boundary conditions are

(a).u(0)=30(b)

(b).u(20)=90

Applying (a) in(1),we get

$U(0)=c_2=30$…………………(2)

Substituting (2) in(1),we get

$U(x)=c_1x+30$…………………(3)

Applying (b) in (3),we get

$U(20)=c_1\times20+30=90$

$C_1=90-30/20=6/2=3$……………………………………….(4)

Substituting(4)in(3), $U(x)=3x+30$

≈≈≈

6.What is the Fourier law of heat conduction.

Ans:

$Q=-KA(U_x)_x$

Q=Quantity of heat flowing

K=thermal conductivity

A=area of cross section

U_X=temperature gradient

(The rate at which heat flows across an area A at distance x from one end of a bar is proportional to temperature gradient.)

≈≈≈

7.State the two-dimensional Laplace equation.

Ans:

$U_{xx}+U_{yy}=0$

≈≈≈

8.In one dimensional heat equation $u_t = \alpha^2U_{xx}$.What does α^2 stands for?

Ans:

α^2=Thermal diffusivity.

≈≈≈

9.Classify the partial differential equation $3u_{xx}+4u_{xy}+3u_y-2u_x=0$

Ans:

Given $3u_{xx}+4u_{xy}+3u_y-2u_x=0$

A=3,B=4,C=0

$B^2-4AC=16>0$

p.d.e is hyperbolic.

≈≈≈

10.Write the one dimensional wave equation with initial and boundary conditions in which the initial position of the string is f(x) and the initial velocity imparted at each point x is g(x).

Ans:

The one dimensional wave equation is $U_{tt}=\alpha^2 U_{xx}$

The boundary conditions are (i)u(0,t)=0

(ii)u(x,0)=f(x)

(iii)u(l,t)=0

(iv)u_t (x,0)=g(x)

≈≈≈

11 .What is the basic difference between the solutions of one dimensional wave equation and one dimensional heat equation.

Ans:

Solution of the one dimensional wave equation is of periodic in nature.

But Solution of the one dimensional heat equation is not of periodic in nature.

≈≈≈

12.In steady state conditions derive the solution of one dimensional heat flow equation.

Ans:

When steady state conditions exist the heat flow equation is independent of time t.

$$U_t=0$$

The heat flow equation becomes

$$U_{xx}=0$$

≈≈

13.In the wave equation $U_{tt}= c^2U_{xx}$,what does c^2 stand for?

Ans:

C^2=T/m=Tension/mass per unit length

≈≈

14.Classify the partial differential equation $U_{xx}+ 2U_{xy}+U_{yy} = e^{(2x+3y)}$

Ans:

A=1, B=2, C=1

$\Delta=B^2-4AC$

$= 4-4 =0$

p.d.e is parabolic.

≈≈

15.In 2D heat equation or Laplace equation ,What is the basic assumption.

Ans:

When the heat flow is along curves instead of straight lines,the curves lying in parallel planes the flow is called two dimensional.

16.Define steady state temperature distribution.

Ans:

If the temperature will not change when time varies is called steady state temperature distribution.

≈≈≈

17.State any two laws which are assumed to derive one dimensional heat equation.

Ans:

(i)The sides of the bar are insulated so that the loss or gain of heat from the sides by conduction or radiation is negligible.

(ii)The same amount of heat is applied at all points of the face.

≈≈≈

18.Classify the following partial differential equations.

(a)$y^2U_{xx}-2xyU_{xy}+x^2U_{YY}+2U_x-3U=0$

(b)$y^2U_{xx}+U_{yy}+U_x^{\,2}+U_y^{\,2}+7 =0$

Ans:

(a)$A=y^2$, $B=-2xy$,$C=x^2$

$B^2-4AC=4x^2y^2-4x^2y^2 \;=0$

p.d.f is parabolic.

(b) $A=y^2$,$B=0$,$C=1$

$B^2-4AC=-4y \;<0$

p.d.f is Elliptic.

≈≈≈

19.Classify the following second order partial differential equation

(a)$4U_{xx}+4U_{xy}+U_{yy} -6U_x-8U_y-16U=0$

(b)$U_{XX} + U_{yy} = U^2_x + U^2_y$

Ans;

(a)A=4,B=4,C=1

B^2-4AC=0

p.d.e is parabolic equation.

(b) A=1,B=0,C=1

B^2-4AC=-4

<0

p.d.e is Elliptic equation.

≈≈

20.The ends A and B of a rod of length 10 cm long have their temperature kept at 20˚C and 70˚C. Find the steady state temperature distribution on the rod.

Ans:

When steady state conditions exists the heat flow equation is

$U_{xx}=0$

i.e., $u(x)=c_1x + c_2$……………(1)

The boundary conditions are (a) u(0)=20 (b) u(10)=70

Applying (a) in (1) , we get

$U(0)=c_2=20$

Substituting $c_2=20$ in (1) ,we get

$U(x) = c_1x+20$ ………………(2)

Applying (b) in (2) , we get

$U(10)=c_1 10+20=70$

Substituting c_1=5 in (2) , we get

U(x)=5x+20

≈≈≈

21. Classify the partial differential equation $U_{xx}+xU_{yy}=0$

Ans:

Here A=1 ,B=0 ,C=x

$B^2-4AC= -4x$

(i)Elliptic if x>0

(ii)Parabolic if x=0

(iii)Hyperbolic if x<0

≈≈≈

22. An insulated rod of length l =60 cm has its ends at A and B maintained at 30°C and 40°C respectively. Find the steady state solution.

Ans:

The heat flow equation is $u_t= \alpha^2 u_{xx}$ ………………(1)

When steady state condition exist the heat flow equation becomes

$U_{xx}=0$

i.e $U_{xx}=0$

$u(x)=c_1x+c_2$ ………………….(2)

The end conditions are

U(0)=30 …………………(3)

U(l)=40 …………………(4)

Substituting (3) in (2) we get

$U(0)=c_2=30$

$U(x)=c_1x+30$……………………(5)

Substituting (4) in(5) we get

$U(l)=c_1l+30=40$

$C_1l=40$

$C_1=40/l$………………………(6)

Substituting (6)in (5) we get

$U(x)=40x/ l+30$

≈≈≈

23.Write the solution of one dimensional heat flow equation , when the time derivative is absent.

Ans:

When time derivative is absent the heat flow equation is $U_{xx}=0$.

≈≈≈

24. If the solution of one dimensional heat flow equation depends neither on Fourier cosine series nor on Fourier sine series , what would have been the nature of the end conditions.

Ans:
One end should be thermally insulated and the other end is at zero temperature.

≈≈≈

25.Explain the initial and boundary value problems .

Ans:

In ordinary differential equations, first we get the general solution which contains the arbitrary constants and then we determine these constants from the given initial values.This type of problems are called initial value problems.

In many physical problems, we always seek a solution of the differential equations, Whether it is ordinary or partial, which satisfies some specified conditions called boundary conditions. Any differential equations together with these boundary Conditions is called boundary value problems.

UNIT – V

Z-TRANSFORMS AND DIFFERENCE EQUATIONS

1. Define Z-tranforms of the sequence {x(n)}.
   Ans:

   a) Z-transform (two sided or bilateral) :
   Let {x(n)} be a sequence defined for all integers then its Z-transform is defined to be

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

   where Z is an arbitrary complex number.

   b) Z-transform (one-sided or unilateral) :
   Let {x(n)} be a sequence defined for n=0,1,2,...and x(n)=0 for n<0 , then its Z-transform is defined to be

$$Z\{x(n)\} = X(Z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

   where Z is an arbitrary complex number.

~~~~~~

2. Define Z-tranforms of f(t).
   Ans:
   Z-transform for discrete values of t :
   If f(t) is a function defined for discrete values of t where t=nT , n=0,1,2,...T being the sampling period , then Z-transform of f(t) is defined as

$$Z\{f(t)\} = F(Z) = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

~~~~~~

3. Prove that $Z[a^n] = \frac{z}{z-a}$ $if\ |z| > |a|$.

   Solution:

   We know that $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$

   Here x(n) = a^n

$$\therefore Z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n \frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{a}{z}\right]^n$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= \left[1 - \frac{a}{z}\right]^{-1} \qquad [\because\ (1-x)^{-1} = 1 + x + x^2 + \cdots]$$

$$= \left[\frac{z-a}{z}\right]^{-1}$$

$$= \frac{z}{z-a} \quad , |z| > |a|$$

ie., $Z[a^n] = \frac{z}{z-a}\ \ if\ \ |z| > |a|$.

~~~~~~

4. State and prove initial value theorem in Z-transform.

   Statement:

   If $\{f(t)\} = F(z)$, then $f(0) = \lim_{z\to\infty} F(z)$.

   Proof:

   WKT ,

$$Z\{f(t)\} = F(Z) = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$= f(0T) + f(1T)z^{-1} + f(2T)z^{-2} + \cdots$$

$$= f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \cdots$$

$$\therefore\ \lim_{z\to\infty} F(z) = \lim_{z\to\infty}\left[f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \cdots\right]$$

$$= f(0)$$

   ie., $f(0) = \lim_{z\to\infty} F(z)$.

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5. State First Shifting theorem.

   Statement:

   (i) If $Z\{f(t)\} = F(z)$, then $Z\{e^{-at}f(t)\} = F[ze^{aT}]$

   (ii) If $Z\{f(t)\} = F(z)$, then $Z\{e^{at}f(t)\} = F[ze^{-aT}]$

   (iii) If Z$\{f(t)\} = F(z)$, then $Z\{a^n f(t)\} = F\left[\frac{z}{a}\right]$

   (iv) If $Z\{f(n)\} = F(z)$, then $Z\{a^n f(n)\} = F\left[\frac{z}{a}\right]$

~~~~~~

6. Find $Z[a^n n]$

   Solution:

   W.K.T. $Z[a^n f(n)] = F\left[\frac{z}{a}\right]$

   Here f(n) = n

   $\therefore\ Z[a^n n] = [Z[n]]_{z\to\frac{z}{a}}$
~~~~~~

$$= \left[\frac{z}{(z-1)^2}\right]_{z \to \frac{z}{a}}$$

$$= \frac{\left(\frac{z}{a}\right)}{\left(\frac{z}{a}-1\right)^2}$$

$$= \frac{\left(\frac{z}{a}\right)}{\left(\frac{z-a}{a}\right)^2}$$

$$= \frac{\left(\frac{z}{a}\right)}{\frac{(z-a)^2}{a^2}}$$

$$= \frac{az}{(z-a)^2}$$

$$ie., Z[a^n n] = \frac{az}{(z-a)^2}$$

~~~~~~

7. State the Differentiation in the Z-Domain.

   Statement:

   (i) $Z[nf(t)] = -z\frac{d}{dz}F[z]$

   (ii) $Z[nf(n)] = -z\frac{d}{dz}F[z]$

~~~~~~

8. Find $Z[n^2]$

   Solution:

W.K.T. $Z[nf(n)] = -z\frac{d}{dz}F[z]$

$$Z[n^2] = Z[nn] = -z\frac{d}{dz}\, Z[n]$$

$$= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] \qquad \left[\because Z[n] = \frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2(1) - z[2(z-1)]}{(z-1)^4}\right]$$

$$= -z\left[\frac{z-1-2z}{(z-1)^3}\right]$$

$$= z\left[\frac{(z+1)}{(z-1)^3}\right]$$

$$= \frac{z^2+z}{(z-1)^3}$$

$$ie., Z[n^2] = \frac{z^2+z}{(z-1)^3}$$

9. Find the Z-transform of (n+1)(n+2).

Solution:

$$Z[(n+1)(n+2)] = Z[n^2 + 2n + n + 2]$$
$$= Z[n^2 + 3n + 2]$$
$$= Z[n^2] + 3Z[n] + 2Z[1]$$
$$= \left[\frac{z^2+z}{(z-1)^3}\right] + 3\left[\frac{z}{(z-1)^2}\right] + 2\left[\frac{z}{z-1}\right]$$
$$= \frac{(z^2+z)+3z(z-1)+2z(z-1)^2}{(z-1)^3}$$

~~~~~~

10. State and prove Second Shifting theorem.

Statement:

If $\{f(t)\} = F(z)$, then $Z[f(t+T)] = zF(z) - zf(0)$.

Proof:

W.K.T., $Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$

$$\therefore Z[f(t+T)] = \sum_{n=0}^{\infty} f(nT+T) z^{-n}$$
$$= \sum_{n=0}^{\infty} f[(n+1)T] z^{-n}$$
$$= \frac{z}{z}\sum_{n=0}^{\infty} f[(n+1)T] z^{-n}$$
$$= z\sum_{n=0}^{\infty} f[(n+1)T] \frac{z^{-n}}{z}$$
$$= z\sum_{n=0}^{\infty} f[(n+1)T] z^{-n}\ z^{-1}$$
$$= z\sum_{n=0}^{\infty} f[(n+1)T] z^{-(n+1)}$$
$$= z\sum_{m=1}^{\infty} f(mT) z^{-m} \qquad \text{where m=n+1}$$
$$= z\left[\sum_{m=0}^{\infty} f(mT) z^{-m} - f(0)\right]$$
$$= zf(z) - zf(0)$$

ie., $Z[f(t+T)] = zF(z) - zf(0)$.

~~~~~~

11. Prove that $Z[f(n+1)] = zF(z) - zf(0)$.

Proof:

W.K.T., $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$Z\{f(n+1)\} = \sum_{n=0}^{\infty} f(n+1) z^{-n}$$
$$= \frac{z}{z}\sum_{n=0}^{\infty} f(n+1) z^{-n}$$
$$= z\sum_{n=0}^{\infty} f(n+1) \frac{z^{-n}}{z}$$
$$= z\sum_{n=0}^{\infty} f(n+1) z^{-n} z^{-1}$$

$$= z\sum_{n=0}^{\infty} f(n+1)z^{-(n+1)}$$
$$= z\sum_{m=1}^{\infty} f(m)z^{-m} \qquad \text{where m=n+1}$$
$$= z[\sum_{m=0}^{\infty} f(m)z^{-m} - f(0)]$$
$$= zf(z) - zf(0)$$

ie., $Z[f(n+1)] = zF(z) - zf(0).$

~~~~~~

12. Find the Z-transform of unit sample sequence.

Solution:

W.K.T., $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)z^{-n}$

Also W.K.T., $\delta(n)$ is the unit sample sequence.

ie., $\delta(n) = \begin{cases} 1 & for\ \ n=0 \\ 0 & for\ \ n>0 \end{cases}$ ....(1)

Now, $Z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n)z^{-n}$

$= 1 + 0 + 0 + \cdots \qquad [\because (1)]$

$= 1$

$ie., Z\{\delta(n)\} = 1$

~~~~~~

13. Find the Z-transform of unit step sequence.

Solution:

W.K.T., $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)z^{-n}$

Also W.K.T., $\delta(n)$ is the unit step sequence.

ie., $u(n) = \begin{cases} 1 & for\ \ n \geq 0 \\ 0 & for\ \ n<0 \end{cases}$ ....(1)

Now, $Z[u(n)] = \sum_{n=0}^{\infty} u(n)z^{-n}$

$= \sum_{n=0}^{\infty} z^{-n} \qquad [\because (1)]$

$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$

$= \left[1 - \frac{1}{z}\right]^{-1}$

$= \left[\frac{z-1}{z}\right]^{-1}$

$= \frac{z}{z-1}$

14. State Final Value theorem.

Statement:

If $\{f(t)\} = F(z)$, then $\lim_{t\to\infty} f(t) = \lim_{z\to 1}(z-1)\,F(z)$.

~~~~~~

15. State Convolution theorem on Z-transform.

Statement:

(i) If Z[x(n)] = X(z) and Z[y(n)] = Y(z) then

Z{x(n)*y(n)} = X(z).Y(z)

(ii) Z[f(t)] = F(z) and Z[g(t)] = G(z) then

Z{f(t)*g(t)} = F(z).G(z)

~~~~~~

16. Find $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

Solution:

$$z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-b}\right]$$

$$= z^{-1}\left[\frac{z}{z-a}\right]\cdot z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n$$

$$= \sum_{m=0}^{n} a^m\, b^{n-m}$$

$$= \sum_{m=0}^{n} a^m\, b^n\, b^{-m}$$

$$= b^n \sum_{m=0}^{n} a^m \frac{1}{b^m}$$

$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n\left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right]$$

$$= b^n\left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\left(\frac{a}{b} - 1\right)}\right] \qquad \left[\because 1 + a + a^2 + \cdots + a^{n-1} = \frac{a^n - 1}{a-1}\right]$$

$$= b^n\left[\frac{\left(\frac{a^{n+1}}{b^{n+1}} - 1\right)}{\left(\frac{a-b}{b}\right)}\right]$$

$$= b^n\left[\frac{\left(\frac{a^{n+1} - b^{n+1}}{b^{n+1}}\right)}{\left(\frac{a-b}{b}\right)}\right]$$

$$= b^n\left[\frac{\left(\frac{a^{n+1} - b^{n+1}}{b^n}\right)}{(a-b)}\right]$$

$$= \frac{a^{n+1}-b^{n+1}}{(a-b)}$$

$$ie., z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{a^{n+1}-b^{n+1}}{(a-b)}$$

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17. Form a difference equation by eliminating arbitrary constant from $u_n = a\,2^{n+1}$

Solution:

Given, $u_n = a\,2^{n+1}$ ......(1)

$u_{n+1} = a\,2^{n+2}$

$= a\,2^{n+1}\,2$

$= 2a2^{n+1}$ ....(2)

Eliminating the constant 'a' , we get,

$$\begin{vmatrix} u_n & 1 \\ u_{n+1} & 2 \end{vmatrix} = 0$$

$2u_n - u_{n+1} = 0$

~~~~~~

18. Form the difference equation from $y_n=a+b3^n$

Solution:

Given, $y_n=a+b3^n$ ......(1)

$Y_{n+1}=a+b3^{n+1}$

$=a+3b\,3^n$ ....(2)

$Y_{n+2}=a+b3^{n+2}$

$=a+9b\,3^n$ ....(3)

Eliminating a and b from (1),(2)&(3) we get,

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 3 \\ y_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$Y_n[9-3]-(1)[9y_{n+1}-3y_{n+2}]+(1)[y_{n+1}-y_{n+2}] = 0$

$6y_n-9y_{n+1}+3y_{n+2}+y_{n+1}-y_{n+2} = 0$

$2y_{n+2}-8y_{n+1}+6y_n = 0$

$y_{n+2}-4y_{n+1}+3y_n = 0$

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19. Find $Z\left[\frac{a^n}{n!}\right]$ in Z-transform.

Solution:

W.K.T., $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$
~~~~~~

$$\therefore\ Z\left[\frac{a^n}{n!}\right] = \sum_{n=0}^{\infty}\left[\frac{a^n}{n!}\right] z^{-n}$$

$$= \sum_{n=0}^{\infty}\frac{(az^{-1})^{-n}}{n!}$$

$$= 1 + \frac{az^{-1}}{1!} + \frac{(az^{-1})^2}{2!} + \cdots$$

$$= e^{az^{-1}} \qquad \left[\because\ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right]$$

$$= e^{a/z}$$

$$ie., Z\left[\frac{a^n}{n!}\right] = e^{a/z}$$

~~~~~~

20. Find $Z\left[e^{-iat}\right]$ using Z-transform.

Solution:

$$Z\left[e^{-iat}\right] = Z\left[e^{-iat}.1\right]$$

$$= \{z[1]\}_{z\to ze^{iaT}} \qquad \text{[By Shifting property]}$$

$$= \left[\frac{z}{z-1}\right]_{z\to ze^{iaT}} \qquad \left[\because Z[1] = \frac{z}{z-1}\right]$$

$$= \frac{ze^{iaT}}{ze^{iaT}-1}$$

$$ie., Z\left[e^{-iat}\right] = \frac{ze^{iaT}}{ze^{iaT}-1}$$

~~~~~~

21. Find the Z-Transform of n.

Solution:

W.K.T., $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)z^{-n}$

Here $x(n) = n$

$$Z[n] = \sum_{n=0}^{\infty} nz^{-n}$$

$$= \sum_{n=0}^{\infty}\frac{n}{z^n}$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right]$$

$$= \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2} \qquad \left[\because\ (1-x)^{-2} = 1 + 2x + 3x^2 + \cdots\right]$$

$$= \frac{1}{z}\left[\frac{z-1}{z}\right]^{-2}$$

$$= \frac{1}{z}\left[\frac{z}{z-1}\right]^{2}$$

$$= \frac{1}{z}\frac{z^2}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}$$

$$ie., Z[n] = \frac{z}{(z-1)^2}$$

~~~~~~

22. Find the Z-Transform of cosnθ and sinnθ .

Solution:

W.K.T., $Z[a^n] = \frac{z}{z-a}$

$$\therefore Z\left[\left(e^{i\theta}\right)^n\right] = \frac{z}{z-e^{i\theta}}$$

$$\Longrightarrow \quad Z[cosn\theta + isinn\theta] = \frac{z}{z-[cos\theta+isin\theta]}$$

$$= \frac{z}{z-cos\theta-isin\theta}$$

$$= \frac{z}{(z-cos\theta)-isin\theta} \times \frac{(z-cos\theta)+isin\theta}{(z-cos\theta)+isin\theta}$$

$$= \frac{z[(z-cos\theta)+isin\theta]}{(z-cos\theta)^2+sin^2\theta}$$

$$= \frac{z(z-cos\theta)}{(z-cos\theta)^2+sin^2\theta} + i\frac{zsin\theta}{(z-cos\theta)^2+sin^2\theta}$$

Equating the real and imaginary parts on both sides, we get,

$$Z[cosn\theta] = \frac{z(z-cos\theta)}{(z-cos\theta)^2+sin^2\theta}$$

$$Z[sinn\theta] = \frac{zsin\theta}{(z-cos\theta)^2+sin^2\theta}$$

~~~~~~

23. Find $Z\left[\frac{1}{n}\right], n > 0.$

Solution:

W.K.T., $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)z^{-n}$

Here $x(n) = \frac{1}{n}$

$$\therefore Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty}\frac{1}{n}z^{-n}$$

$$= \sum_{n=1}^{\infty}\frac{1}{nz^n}$$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots$$

$$= \frac{1}{z} + \frac{\left(1/z\right)^2}{2} + \frac{\left(1/z\right)^3}{3} + \cdots$$

$$= -log\left(1-\frac{1}{z}\right) \qquad \left[\because \log(1-x) = -x-\frac{x^2}{2}-\cdots\right]$$

$$= -log\left(\frac{z-1}{z}\right)$$

$$= log\left(\frac{z-1}{z}\right)^{-1} \qquad [\because \log a^p = p\ log a]$$

$$= log\left(\frac{z}{z-1}\right)$$

$$ie., Z\left[\frac{1}{n}\right] = log\left(\frac{z}{z-1}\right)$$

~~~~~~

24. Find the inverse Z-transform of $\frac{z^2}{(z-a)^2}$ using convolution theorem.

Solution:

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\left(\frac{z}{z-a}\right)\cdot\left(\frac{z}{z-a}\right)\right]$$

$$= a^n * a^n \qquad where * denotes\ convolution.$$

$$= \sum_{k=0}^{n} a^k\ a^{n-k}$$

$$= \sum_{k=0}^{n} a^k\ a^n a^{-k}$$

$$= \sum_{k=0}^{n} a^n$$

$$= a^n \sum_{k=0}^{n}(1)$$

$$= a^n[1+1+1+\cdots(n+1)times]$$

$$= a^n + a^n + a^n + \cdots[(n+1)times]$$

$$= (n+1)a^n$$

$$ie., Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = (n+1)a^n$$

~~~~~~

25. Evaluate $Z^{-1}\left[\frac{z}{z^2+7z+10}\right]$.

Solution:

$$Let\ X(z) = \frac{z}{(z+5)(z+2)}$$

$$\frac{X(z)}{z} = \frac{1}{(z+5)(z+2)} = \frac{A}{(z+2)} + \frac{B}{(z+5)}$$

$$\Longrightarrow \quad 1 = A(z+5) + B(z+2)$$

$Put\ z = -2\ , we\ get$

$$1 = A(-2+5) + B(0)$$

$$1 = 3A$$

$$A = \frac{1}{3}$$

$Put\ z = -5\ ,we\ get$

$$1 = A(0) + B(-5+2)$$

$$1 = -3B$$

$$B = \frac{-1}{3}$$

$$\therefore \frac{X(z)}{z} = \frac{1}{3(z+2)} - \frac{1}{3(z+5)}$$

$$X(z) = \frac{z}{3(z+2)} - \frac{z}{3(z+5)}$$

$$\Longrightarrow Z\{x(n)\} = \frac{z}{3(z+2)} - \frac{z}{3(z+5)}$$

$$\Longrightarrow x(n) = \frac{1}{3}Z^{-1}\left[\frac{z}{(z+2)}\right] - \frac{1}{3}Z^{-1}\left[\frac{z}{(z+5)}\right]$$

$$= \frac{1}{3}(-2)^n - \frac{1}{3}(-5)^n$$

$$= \frac{1}{3}[(-1)^n(2^n - 5^n)]$$

$$= \frac{(-1)^n}{3}[2^n - 5^n]$$

$$ie., Z^{-1}\left[\frac{z}{z^2+7z+10}\right] = \frac{(-1)^n}{3}[2^n - 5^n]$$

~~~~~~
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