Solution of Difference equations

The following expressions are mainly used in solving difference equation using Z-transform

$$Z[y_{n+1}] = zY(z) - zy(0)$$

$$Z[y_{n+2}] = z^2 Y(z) - z^2 y(0) - zy(1)$$

① Solve $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0$; $y_1 = 1$ using Z-transform method.

Let $Y(z) = Z[y_n]$

Given $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$

Taking Z-transform on both sides

$$z^2 Y(z) - z + 4zY(z) + 3Y(z) = \frac{z}{z-2}$$

$$(z^2 + 4z + 3)\, Y(z) = \frac{z}{z-2} + z = \frac{z + z^2 - 2z}{z-2}$$

$$Y(z) = \frac{z^2 - z}{(z-2)(z+1)(z+3)}$$

$$\frac{Y(z)}{z} = \frac{z-1}{(z-2)(z+1)(z+3)} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{C}{z+3}$$

$$z - 1 = A(z+1)(z+3) + B(z-2)(z+3) + C(z-2)(z+1)$$

Put $z = -1$

$-2 = -6B$

$\boxed{B = 1/3}$

Put $z = 2$

$1 = 15A$

$\boxed{A = 1/15}$

Put $z = -3$

$-4 = 10C$

$\boxed{C = -2/5}$

$$\frac{Y(z)}{z} = \frac{1}{15}\frac{1}{z-2} + \frac{1}{3}\frac{1}{z+1} - \frac{2}{5}\frac{1}{z+3}$$

$$Z^{-1}[Y(z)] = \frac{1}{15} Z^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{3} Z^{-1}\left[\frac{z}{z+1}\right] - \frac{2}{5} Z^{-1}\left[\frac{1}{z+3}\right]$$

$$= \frac{1}{15} 2^n + \frac{1}{3}(-1)^n - \frac{2}{5}(-3)^n$$

2) solve using Z-transform $y(n+2) - 4y(n+1) + 4y(n) = 0$ with $y_0 = 1$ and $y_1 = 0$.

Let $Y(z) = Z[y_n]$

Given $y(n+2) - 4y(n+1) + 4y(n) = 0$

Taking Z-transform on both sides

$$z^2 Y(z) - z^2 y(0) - z y(1) - 4[z Y(z) - z y(0)] + 4 Y(z) = 0$$

$$z^2 Y(z) - z - 4[z Y(z)] + 4 Y(z) = 0$$

$$(z^2 - 4z + 4) Y(z) = z$$

$$Y(z) = \frac{z}{z^2 - 4z + 4} = \frac{z}{(z-2)^2}$$

$$\frac{Y(z)}{z} = \frac{1}{(z+2)^2} = \frac{A}{z+2} + \frac{B}{(z-2)^2}$$

$$1 = A(z-2) + B$$

Put $z = 2$

$\boxed{1 = B}$

Put $z = 0$

$$1 = -2A + B$$

$$-2A + 1 = 1$$

$z = 2$ is a pole of order 2.

$$R_1 = \lim_{z \to 2} \frac{d}{dz} \frac{z}{\cancel{(z-2)^2}} \cancel{(z-2)^2} z^{n-1}$$

$$= \lim_{z \to 2} \frac{d}{dz} z^n = \lim_{z \to 2} n z^{n-1}$$

$$= n (2)^{n-1}$$

$$\therefore Z^{-1} \{ Y(z) \} = n 2^{n-1}$$