Applications of Partial Differential Equation

Introduction: Nature of PDE

The 2nd order PDE is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f(x, y, u, u_x, u_y) = 0 \rightarrow (1)$$

(or)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + f(x, y, u, u_x, u_y) = 0$$

Eqn (1) is classified as,

i). $B^2 - 4AC < 0 \rightarrow$ Elliptic Equation

ii). $B^2 - 4AC = 0 \rightarrow$ Parabolic Equation

iii). $B^2 - 4AC > 0 \rightarrow$ Hyperbolic Equation.

Problems:

1. classify the equation

$$3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 6\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial x} - u = 0$$

Soln.:

Here $A = 3$, $B = 4$ and $C = 6$

now $B^2 - 4AC = 16 - 72$

$= -56 < 0$

$\therefore$ It is Elliptic

---

2. find the nature of PDE $4u_{xx} + 4u_{xy} + u_{yy} + 2u_x - u_y = 0$

Soln. :

Here $A=4$, $B=4$ and $C=1$

Now, $B^2-4AC = 16-4(4)(1) = 0$

$\therefore$ It is parabolic.

3]. classify the PDE $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial y^2}$

Soln. :

Here $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = 0$

$A=1$, $B=0$ and $C=-1$

Now, $B^2-4AC = 0-4(1)(-1)$
$= 4 > 0$

$\therefore$ It is Hyperbolic

4]. classify the PDE

$x^2 f_{xx} + (1-y^2) f_{yy} = 0$, $-1<y<1$, $-\infty<x<\infty$

Soln.

Here $A=x^2$, $B=0$, $C=1-y^2$.

Now $B^2-4AC = 0-4(x^2)(1-y^2)$
$= 4x^2(y^2-1)$

In $-\infty<x<\infty$, x^2 is always $+ve$

In $-1<y<1$, y^2-1 is $-ve$.

$\therefore$ $B^2-4AC = -ve$ $(x \neq 0)$

$\therefore$ It is Elliptic.

HW 1]. classify $4u_{xx} = u_t$

Scanned by CamScanner

One - dimensional wave Equation [Hyperbolic]

Let $y(x, t)$ represents the one-dimensional wave equation.

i.e., $$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

where $a^2 = \frac{T}{M} = \frac{\text{Tension}}{\text{Mass per unit length}}$.

Possible solutions of ODWE:

1. $y(x,t) = (A_1 e^{px} + A_2 e^{-px})(A_3 e^{pat} + A_4 e^{-pat})$
2. $y(x,t) = (A_5 \cos px + A_6 \sin px)(A_7 \cos pat + A_8 \sin pat)$
3. $y(x,t) = (A_9 x + A_{10})(A_{11} t + A_{12})$

Suitable solution.

$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat)$

Assumptions for deriving ODWE:

1. The motion takes place entirely in one plane i.e., xy-plane.
2. The tension 'T' is constant at all times and at all points of the deflected string.
3. The effect of friction is negligible
4. The string is perfectly flexible.
5. The slope of the deflection curve at all points is neglectable.

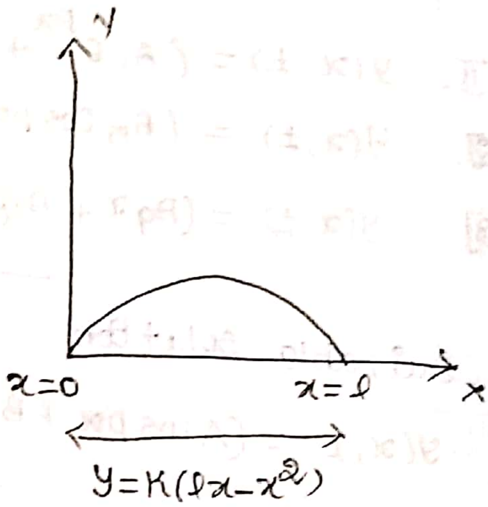
---

①

Type – I

Problems based on vibrating string with zero initial velocity:

1. A string is stretched and fastened at two points $x=0$ and $x=l$, apart. Motion is started by displacing the string into the form $y=K(lx-x^2)$ from which it is released at time $t=0$. Find the displacement of any point on the string at a distance of x from one end at a time t.



Soln. :

The ODWE is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are,

i). $y(0,t)=0, \quad \forall t$

ii). $y(l,t)=0, \quad \forall t$

iii). $\frac{\partial}{\partial t} y(x,0)=0$

iv). $y(x,0) = f(x) = K(lx-x^2)$

The suitable soln. is,

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad \to (1)$$

Applying condition (i) in (1), we get

$\Rightarrow y(0,t) = (A(1) + B(0))\,(C\cos pat + D\sin pat) = 0$

$\Rightarrow A\,(C\cos pat + D\sin pat) = 0$

$\therefore \boxed{A = 0}$ $\quad (\because C\cos pat + D\sin pat \neq 0)$

Subs. $A=0$ in (1),

$$y(x,t) = B\sin px\,(C\cos pat + D\sin pat) \rightarrow (2)$$

Applying condition (ii) in (2), we get

$$y(l,t) = \underbrace{B\sin pl}\,(C\cos pat + D\sin pat) = 0$$

$B \neq 0$ (If $B=0$, then we get a trivial solution)

and $C\cos pat + D\sin pat \neq 0, \ \forall t$

Only possibility, $\sin pl = 0 \Rightarrow pl = \sin^{-1}(0)$

$pl = n\pi, \quad n = 0, 1, \ldots$

$$\boxed{p = \frac{n\pi}{l}}$$

$\sin n\pi = 0$, $\cos n\pi = (-1)^n$

Subs. $p = \frac{n\pi}{l}$ in (2),

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (3)$$

Before applying condition (iii), differentiate (3) partially w.r.t. 't'

$$\frac{\partial}{\partial t}y(x,t) = B\sin\frac{n\pi x}{l}\left[-C\sin\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right) + D\cos\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)\right]$$

Applying condition (iii) we get

$$\frac{\partial}{\partial t}y(x,0) = 0$$

$$B\sin\frac{n\pi x}{l}\left[C(0) + D(1)\frac{n\pi a}{l}\right] = 0$$

$$BD \sin\frac{n\pi x}{l}\ \frac{n\pi a}{l} = 0$$

Here $B \neq 0$ (Already explained)

$\frac{n\pi a}{l} \neq 0$ (All are constants)

$\sin\frac{n\pi x}{l} \neq 0$ (It is defined $\forall x$)

only possibility $\boxed{D=0}$

Subs. $D=0$ in (3), we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\ C\cos\frac{n\pi at}{l}$$

$$= BC \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$= B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

where $B_n = BC$

The most general solution is,

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (4)$$

Applying condition (iv) in (4), we get

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = k(lx - x^2)$$

HRSS $\Rightarrow \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = k(lx-x^2)$ $(\because B_n = b_n)$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

Here $f(x) = k(lx-x^2)$, $0<x<l$

$$\therefore \quad b_n = \frac{2}{l}\int_0^l K(lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2K}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

Bernoulli's: $\int uv\,dx = uv_1 - u'v_2 + u''v_3$

$$= \frac{2K}{l}\left[(lx - x^2)\frac{-\cos\frac{n\pi x}{l}}{n\pi/l} - (l - 2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right) - 0\right]_0^l$$

$$= \frac{2K}{l}\left[-\frac{l}{n\pi}(lx - x^2)\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}(l - 2x)\sin\frac{n\pi x}{l} - \frac{2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2K}{l}\left[\left(0 + 0 - \frac{2l^3}{n^3\pi^3}(-1)^n\right) - \left(0 + 0 - \frac{2l^3}{n^3\pi^3}\right)\right]$$

$$= \frac{2K}{l}\left[-\frac{2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{2K}{l}\,\frac{2l^3}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$b_n = \frac{4Kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$B_n = \begin{cases} \dfrac{8Kl^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Subs. B_n in (A), we get

$$y(x,t) = \sum_{n=odd}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}$$

$$= \frac{8kl^2}{\pi^3} \sum_{n=odd}^{\infty} \frac{1}{n^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}$$