Definition

Let $f(x, y, \alpha) = 0$ be a single parameter family of curves, where α is the parameter. The envelope of this family of curves is a curve which touches every member of the family.

1. Find the envelope of the family of straight lines $y = mx + \dfrac{1}{m}$.

Soln: $y = mx + \dfrac{1}{m} \Rightarrow my = m^2 x + 1$

$\Rightarrow xm^2 - ym + 1 = 0$, which is a quadratic in the parameter *m*.

Here $A = x,\ B = -y, C = 1$

$\therefore$ The envelope is $B^2 - 4AC = 0$

$\Rightarrow \quad y^2 - 4x = 0$

$\Rightarrow \quad y^2 = 4x$.

2. Find the envelope of the family of lines

$y = mx \pm \sqrt{a^2 m^2 - b^2}$, where *m* is the parameter.

Soln: Given the family is $y = mx \pm \sqrt{a^2 m^2 - b^2}$, *m* is the parameter.

$\Rightarrow \quad y - mx = \pm\sqrt{a^2 m^2 - b^2}$

$\Rightarrow (y - mx)^2 = a^2 m^2 - b^2$

$\Rightarrow y^2 - 2mxy + m^2 x^2 = a^2 m^2 - b^2$

$$\Rightarrow m^2(x^2-a^2)-2mxy+(y^2+b^2)=0$$

This is a quadratic in m

Here $A=x^2-a^2,\ B=-2xy,\ C=y^2+b^2$

$\therefore$ The envelope is $B^2-4AC=0$

$$\Rightarrow 4x^2y^2-4(x^2-a^2)(y^2+b^2)=0$$

$$\Rightarrow x^2y^2-(x^2y^2+b^2x^2-a^2y^2-a^2b^2)=0 \qquad [\div \text{ by } 4]$$

$$\Rightarrow b^2x^2-a^2y^2-a^2b^2=0$$

$$\Rightarrow b^2x^2-a^2y^2=a^2b^2$$

$$\Rightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

3. Find the envelope of the straight lines represented by

$x\cos\alpha+y\sin\alpha=a\ \sec\alpha$, where α is a parameter.

Soln: Given $x\cos\alpha+y\sin\alpha=a\ \sec\alpha$

Dividing by $\cos\alpha$, $\quad x+y\tan\alpha=a\ \sec^2\alpha$

$$=a(1+\tan^2\alpha)$$

$$\Rightarrow \quad a\ \tan^2\alpha-y\tan\alpha+(a-x)=0$$

which is a quadratic in $\tan\alpha$

Here $A = a, \quad B = -y, \quad C = a - x$

$\therefore$ The envelope is $B^2 - 4AC = 0 \quad \Rightarrow \quad y^2 - 4a(a-x) = 0,$ which is the envelope.

is the envelope and it is an ellipse.

4. Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters *a* and *b* are related by the equation $a^n + b^n = c^n, c$ being a constant.

Soln: Given $\frac{x}{a} + \frac{y}{b} = 1$ … (1)

$$a^n + b^n = c^n \qquad \text{… (2)}$$

Differentiating (1) and (2) with respect to *a*, treating *b* as a function of *a*,

we get

$(1) \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$ Diff. w.r.to *a*

$$-\frac{x}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0 \Rightarrow \frac{db}{da} = -\frac{b^2 x}{a^2 y} \qquad \text{… (3)}$$

and $(2) \Rightarrow a^n + b^n = c^n$ Diff. w.r.to *a*

$$n\,a^{n-1} + n\,b^{n-1} \cdot \frac{db}{da} = 0 \Rightarrow \frac{db}{da} = -\frac{a^{n-1}}{b^{n-1}} \qquad \text{… (4)}$$

From (3) and (4); $\frac{-b^2 x}{a^2 y} = -\frac{a^{n-1}}{b^{n-1}}$

$$\Rightarrow \quad \frac{x}{a^{n+1}} = \frac{y}{b^{n+1}}$$

$$\Rightarrow \quad \frac{\frac{x}{a}}{a^n} = \frac{\frac{y}{b}}{b^n} = \frac{\frac{x}{a}+\frac{y}{b}}{a^n+b^n} = \frac{1}{c^n}$$ [Using (2)]

$$\therefore \quad \frac{x}{a^{n+1}} = \frac{1}{c^n}$$

$$\Rightarrow \quad a = \left(c^n x\right)^{\frac{1}{n+1}}$$

SImilarly $b = \left(c^n y\right)^{\frac{1}{n+1}}$

Substitute in (1)

$$\frac{x}{c^{n/n+1}\cdot x^{1/n+1}} + \frac{y}{c^{n/n+1}\cdot y^{1/n+1}} = 1$$

$$\Rightarrow \quad x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}}$$

which is the required envelope.