Definition:

The locus of centre of curvature of a given curve is called the evolute of the curve.

The given curve is called an involute of the evolute.

In fact, for the evolute there are many involutes.

1. Find the equation of the evolute of the parabola $y^2 = 4ax$.

Soln: Given $y^2 = 4ax$ … (1)

Let $P(at^2, 2at)$ be any point on the parabola.

Diff. w.r.to x, $2y\frac{dy}{dx} = 4a$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$y_1 = \frac{2a}{y}$$

$$\frac{d^2y}{dx^2} = \frac{-2a}{y^2}\cdot\frac{dy}{dx}$$

$$y_2 = \frac{-4a^2}{y^3}$$

$$\therefore At(at^2, 2at)\ y_1 = \frac{2a}{2at} = \frac{1}{t}$$

$$y_2 = -\frac{4a^2}{(2at)^3} = \frac{-1}{2at^3}$$

$$\therefore y_1 = \frac{1}{t},\ y_2 = -\frac{1}{2at^3}$$

The centre of curvature $(\bar{x},\bar{y})$ at P is given by

$$\bar{x} = x - y_1\left(\frac{1+y_1^{\ 2}}{y_2}\right)$$

$$\bar{x} = at^2 - \frac{\frac{1}{t}\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}} = at^2 + 2a(1+t^2)$$

$$= 3at^2 + 2a$$

$$3at^2 = \bar{x} - 2a$$

$$t^2 = \frac{\bar{x}-2a}{3a} \Rightarrow t = \left(\frac{\bar{x}-2a}{3a}\right)^{1/2} \qquad \ldots(2)$$

$$\bar{y} = y + \left(\frac{1+y_1^{\ 2}}{y_2}\right)$$

$$= 2at + \frac{1+\frac{1}{t^2}}{-\frac{1}{2at^3}} = 2at - 2at(1+t^2) = 2at - 2at - 2at^3$$

$$\bar{y} = -2at^3 \ldots(3)$$

Eliminating t from (2) and (3) we get,

$$\bar{y} = -2a\cdot\left(\frac{\bar{x}-2a}{3a}\right)^{3/2}$$

Squaring both sides, $\bar{y}^2 = \frac{4a^2}{27a^3}\left(\bar{x}-2a\right)^3$

$$27a\bar{y}^2 = 4(\bar{x}-2a)^3$$

$\therefore$ Locus of $(\bar{x}, \bar{y})$ is $27ay^2 = 4(x-2a)^3$,

2. Find the equation of the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Soln: Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ \dots (1)$$

Diff. w.r.to x, we get $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-b^2}{a^2}\cdot\frac{x}{y} \quad \Rightarrow y_1 = \frac{-b^2}{a^2}\frac{x}{y}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-b^2}{a^2}\left[\frac{y\cdot(1) - x\cdot\left(\frac{dy}{dx}\right)}{y^2}\right] = y_2$$

At $(a\cos\theta, b\sin\theta)$, $y_1 = -\frac{b^2}{a^2}\cdot\frac{a\cos\theta}{b\sin\theta}$

$$y_1 = -\frac{b}{a}\frac{\cos\theta}{\sin\theta}$$

$$y_2 = -\frac{b^2}{a^2}\left[\frac{b\sin\theta - a\cos\theta\left(\frac{-b}{a}\frac{\cos\theta}{\sin\theta}\right)}{b^2\sin^2\theta}\right]$$

$$= \frac{-b^2}{a^2}\left[\frac{ab\sin^2\theta + ab\cos^2\theta}{ab^2\sin^3\theta}\right]$$

$$= \frac{-b^2}{a^2} \cdot \frac{ab}{ab^2 \sin^3\theta} = \frac{-b}{a^2} \cdot \frac{1}{\sin^3\theta}$$

$$\therefore y_1 = \frac{-b\cos\theta}{a\sin\theta}, \quad y_2 = \frac{-b}{a^2} \cdot \frac{1}{\sin^3\theta}$$

The centre of curvature $(\bar{x}, \bar{y})$ at P is given by

$$\bar{x} = x - y_1\left(\frac{1 + y_1^2}{y_2}\right)$$

$$= a\cos\theta - \frac{\left(\frac{-b}{a}\frac{\cos\theta}{\sin\theta}\right)\left(1 + \frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right)}{-\frac{b}{a^2}\frac{1}{\sin^3\theta}}$$

$$= a\cos\theta - a\cos\theta\sin^2\theta\left(1 + \frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right)$$

$$= a\cos\theta - a\cos\theta\sin^2\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta\left(1 - \sin^2\theta\right) - \frac{b^2}{a}\cos^3\theta$$

$$\bar{x} = a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$\Rightarrow \bar{x} = \frac{a^2 - b^2}{a}\cos^3\theta \qquad \dots (1)$$

$$\bar{y} = y + \left(\frac{1 + y_1^2}{y_2}\right)$$

$$= b\sin\theta + \frac{1+\dfrac{b^2\cos^2\theta}{a^2\sin^2\theta}}{-\dfrac{b}{a^2\sin^3\theta}}$$

$$= b\sin\theta - \frac{a^2\sin^3\theta}{b}\left(1+\frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right)$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin\theta\cos^2\theta$$

$$= b\sin\theta(1-\cos^2\theta) - \frac{a^2}{b}\sin^3\theta$$

$$= b\sin^3\theta - \frac{a^2}{b}\sin^3\theta$$

$$\Rightarrow \bar{y} = -\frac{a^2-b^2}{b}\sin^3\theta \qquad \ldots (2)$$

Eliminating θ from (1) and (2).

From (1), we get, $\dfrac{a\bar{x}}{a^2-b^2} = \cos^3\theta$

$$\therefore \cos\theta = \left(\frac{a\bar{x}}{a^2-b^2}\right)^{1/3}$$

Similarly, from (2) $\sin\theta = \left(\dfrac{-b\,\bar{y}}{a^2-b^2}\right)^{1/3}$

WKT $\cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow \left[\frac{(a\bar{x})}{a^2-b^2}\right]^{2/3} + \left[\frac{-b\,\bar{y}}{a^2-b^2}\right]^{2/3} = 1$$

$$\Rightarrow \quad \frac{(a\bar{x})^{2/3}}{(a^2-b^2)^{2/3}} + \frac{(-b\,\bar{y})^{2/3}}{(a^2-b^2)^{2/3}} = 1$$

$$\Rightarrow \qquad (a\bar{x})^{2/3} + (-b\,\bar{y})^{2/3} = (a^2-b^2)^{2/3}$$

$\therefore$ Locus of $(\bar{x},\bar{y})$ is $(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$