



UNIT 3- DIFFERENTIAL CALCULUS

Circle of Curvature

Circle of Curvature:

The circle of curvature of a curve $y = f(x)$ at a point $P(x, y)$ with the centre of curvature, $C(\bar{x}, \bar{y})$ and radius curvature ρ is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

1. Find the circle of curvature of the curve $x^3 + y^3 = 3axy$ at the

$$\text{point } \left(\frac{3a}{2}, \frac{3a}{2} \right)$$

Soln :

$$\text{Given } x^3 + y^3 = 3axy \quad \dots (1)$$

Differentiating (1) w.r.to. 'x', we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a \left[x \frac{dy}{dx} + y \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \quad \dots (2)$$

$$y_1 = \left(\frac{dy}{dx} \right)_{\left(\frac{3a}{2}, \frac{3a}{2} \right)}$$

$$= \frac{a \frac{3a}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a \frac{3a}{2}} = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a \frac{3a}{2}}$$

$$= -1$$

$$y_1 = -1$$



UNIT 3- DIFFERENTIAL CALCULUS

Circle of Curvature

$$\text{From (2), } \frac{d^2y}{dx^2} = \frac{(y^2 - ax) \left(a \frac{dy}{dx} - 2x \right) - (ay - x^2) \left(2y \frac{dy}{dx} - a \right)}{(y^2 - ax)^2}$$

$$y_2 = \left(\frac{d^2y}{dx^2} \right)_{\left(\frac{3a}{2}, \frac{3a}{2} \right)}$$

$$= \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2} \right) \left(-a - \frac{6a}{2} \right) - \left(\frac{3a^2}{2} - \frac{9a^2}{4} \right) \left(-\frac{6a}{2} - a \right)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2} \right)^2}$$

$$= \frac{(-4a) - 4a}{\frac{9a^2}{4} - \frac{3a^2}{2}}$$

$$= \frac{-8a}{\frac{3a^2}{4}} = \frac{-32}{3a}$$

$$y_2 = \frac{-32}{3a}$$

∴ Radius of curvature

$$\begin{aligned} \therefore \rho &= \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{[1 + (-1)^2]^{\frac{3}{2}}}{\frac{-32}{3a}} \\ &= \frac{(2)^{\frac{3}{2}} \times 3a}{-32} \\ &= \frac{3\sqrt{2} \times a}{16} \end{aligned}$$

∴ The centre of curvature



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UNIT 3- DIFFERENTIAL CALCULUS

Circle of Curvature

$$\bar{x} = x - y_1 \left(\frac{1 + y_1^2}{y_2} \right)$$

$$= \frac{3a}{2} - \frac{(-1)(2)}{-32} \cdot 3a = \frac{21a}{16}$$

$$= \frac{3a}{2} + \frac{(2)}{-32} \cdot 3a = \frac{21a}{16}$$

∴ The required circle of curvature is

$$\left(x - \frac{21a}{16} \right)^2 + \left(y - \frac{21a}{16} \right)^2 = \left(\frac{3\sqrt{2} \times a}{16} \right)^2$$

$$i.e., (x^2 + y^2) - \frac{21a}{8}(x + y) + \frac{432a^2}{128} = 0$$

2. Find the circle of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point $\left(\frac{a}{4}, \frac{a}{4} \right)$.

Soln:

$$\text{Given curve: } \sqrt{x} + \sqrt{y} = \sqrt{a} \text{ ----> (1)}$$

$$\text{Point: } P \left(\frac{a}{4}, \frac{a}{4} \right)$$

Centre of curvature $C(\bar{x}, \bar{y})$,

$$\bar{x} = x - y_1 \left(\frac{1 + y_1^2}{y_2} \right)$$

$$\bar{y} = y + \left(\frac{1 + y_1^2}{y_2} \right)$$



UNIT 3- DIFFERENTIAL CALCULUS

Circle of Curvature

From (1),

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \Rightarrow \text{Differentiate w.r.to. } x$$

$$\frac{d}{dx}(\sqrt{x} + \sqrt{y}) = \frac{d}{dx}(\sqrt{a})$$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-1/2}}{y^{-1/2}} = -\frac{y^{1/2}}{x^{1/2}}$$

$$\text{at } P\left(\frac{a}{4}, \frac{a}{4}\right), \frac{dy}{dx} = -1$$

$$y_1 = -1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{-y^{1/2}}{x^{1/2}}\right)$$

$$= -\left[\frac{x^{1/2}\left(\frac{1}{2}y^{-1/2}\right)\left(\frac{dy}{dx}\right) - y^{-1/2}\left(\frac{1}{2}x^{-1/2}\right)}{x}\right]$$

$$= -\frac{1}{2}\left[\frac{x^{1/2}y^{-1/2}\frac{dy}{dx} - y^{-1/2}x^{-1/2}}{x}\right]$$



UNIT 3- DIFFERENTIAL CALCULUS

Circle of Curvature

$$\begin{aligned} \text{At } P\left(\frac{a}{4}, \frac{a}{4}\right), \quad \frac{d^2y}{dx^2} &= \frac{-1}{2} \left[\frac{(1)(-1) - (1)}{\frac{a}{4}} \right] \\ &= \frac{-1}{2} \left[\frac{-2}{\frac{a}{4}} \right] \end{aligned}$$

$$y_2 = \frac{4}{a}$$

Consider $\frac{1+y_1^2}{y_2} = \frac{1+(-1)^2}{\frac{4}{a}} = \frac{2}{\frac{4}{a}} = \frac{a}{2}$

$$\bar{x} = \frac{a}{4} - (-1)\left(\frac{a}{2}\right) = \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}$$

$$\bar{y} = \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}$$

We know that

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\begin{aligned} \therefore \text{At } P\left(\frac{a}{4}, \frac{a}{4}\right), \quad \rho &= \frac{[1+(-1)^2]^{3/2}}{\frac{4}{a}} = (2)^{3/2} \times \frac{a}{4} \\ &= 2\sqrt{2} \times \frac{a}{4} \end{aligned}$$



UNIT 3- DIFFERENTIAL CALCULUS

Circle of Curvature

$$= \frac{a\sqrt{2}}{2}$$

$$\therefore \rho = \frac{a\sqrt{2}}{2}$$

From (2), we have

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \left(\frac{a}{\sqrt{2}}\right)^2,$$

3. Find the equation of the circle of curvature at (c, c) on $xy = c^2$.

Soln:

Given curve: $xy = c^2$ ----> (1)

Point: $P(c, c)$

Centre of curvature $C(\bar{x}, \bar{y})$,

$$\bar{x} = x - y_1 \left(\frac{1 + y_1^2}{y_2} \right)$$

$$\bar{y} = y + \left(\frac{1 + y_1^2}{y_2} \right) \text{ where } y_1 = \frac{dy}{dx}; y_2 = \frac{d^2y}{dx^2}.$$

From (1),

$$xy = c^2$$

$$\frac{d}{dx}(xy) = \frac{d}{dx}(c^2)$$

$$x \frac{dy}{dx} + y(1) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$



UNIT 3- DIFFERENTIAL CALCULUS

Circle of Curvature

$$\text{At } P(c, c), \frac{dy}{dx} = \frac{-c}{c} = -1$$

$$y_1 = -1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$= \frac{d}{dx} \left(\frac{-y}{x} \right)$$

$$= - \left[\frac{x \frac{dy}{dx} - y(1)}{x^2} \right]$$

$$\begin{aligned} \therefore \text{At } P(c, c), \frac{d^2y}{dx^2} &= - \left[\frac{c(-1) - (c)}{c^2} \right] \\ &= \frac{2c}{c^2} \end{aligned}$$

\therefore

$$y_2 = \frac{2}{c}$$

$$\text{Consider } \frac{1+y_1^2}{y_2} = \frac{1+(-1)^2}{\frac{2}{c}} = \frac{2}{\frac{2}{c}} = c$$

$$\bar{x} = c - (-1)(c) = 2c$$

$$\bar{y} = c + c = 2c$$

We know that

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$



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$$\rho = \frac{(1 + (-1)^2)^{3/2}}{\frac{2}{c}} = (2)^{3/2} \times \frac{c}{2}$$

\therefore at $P(c, c)$,

$$= 2\sqrt{2} \times \frac{c}{2}$$

$$= c\sqrt{2}$$

$$\therefore \rho = c\sqrt{2}$$

\therefore (2) becomes

$$(x - 2c)^2 + (y - 2c)^2 = (c\sqrt{2})^2 .$$