Centre of Curvature:

The centre of curvature of a curve $y = f(x)$ at a point $P(x, y)$ is denoted by $C(\bar{x}, \bar{y})$ and

$$\bar{x} = x - y_1\left(\frac{1+y_1^{\,2}}{y_2}\right) \quad \text{and}$$

$$\bar{y} = y + \left(\frac{1+y_1^{\,2}}{y_2}\right)$$

1. Find the centre of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (a,b).

Soln:

Given curve: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ -----> (1)

Point: $P(a,b)$

Differentiating (1) w.r.t 'x', we get

$$\frac{1}{a^2}2x+\frac{1}{b^2}2y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-x}{y}\cdot\frac{b^2}{a^2} \qquad \text{----->} \quad (2)$$

$$\therefore \frac{dy}{dx} \text{ at } (a,b) = \frac{-a}{b}\cdot\frac{b^2}{a^2}=\frac{-b}{a}$$

$$y = \frac{-b}{a}$$

From (2),

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$= \frac{d}{dx}\left[\frac{-x}{y}\cdot\frac{b^2}{a^2}\right]$$

$$= \frac{-b^2}{a^2}\frac{d}{dx}\left[\frac{x}{y}\right]$$

i.e., $$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left[\frac{y(1)-x\frac{dy}{dx}}{y^2}\right]$$

$$\therefore \frac{d^2y}{dx^2} \text{ at } P(a,b) = \frac{-b^2}{a^2}\left[\frac{b-a\left(\frac{-b}{a}\right)}{b^2}\right]$$

$$y_2 = \frac{2b}{a^2}$$

Consider $$\frac{1+y_1^{\,2}}{y_2} = \frac{1+\left(\frac{-b}{a}\right)^2}{\frac{-2b}{a^2}} = \frac{a^2+b^2}{-2b} = \frac{-\left(a^2+b^2\right)}{+2b}$$

$$\bar{x} = a - \left(\frac{-b}{a}\right)\left(\frac{a^2+b^2}{2b}\right) = \frac{2a^2-a^2-b^2}{2a} = \frac{a^2-b^2}{2a}$$

$$\bar{y} = b - \left(\frac{a^2+b^2}{2b}\right) = \frac{2b^2-a^2-b^2}{2b} = -\left(\frac{a^2-b^2}{2b}\right)$$

$\therefore$ Centre of curvature $C(\bar{x},\bar{y})$ at $P(a,b)$ is $\bar{x} = \dfrac{a^2+b^2}{2a}$, $\bar{y} = \dfrac{b^2-a^2}{2b}$.

2. Find the centre of curvature of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec\theta, b\tan\theta)$.

Soln:

Given curve: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ -----> (1)

Point: $P(a\sec\theta, b\tan\theta)$

Here given point is the parametric representation of (1).

$\therefore x = a\sec\theta,$ -----> (2)

$y = b\tan\theta,$ -----> (3) where θ is the parameter.

$$\frac{dx}{d\theta}=a\sec\theta\tan\theta,\quad \frac{dy}{d\theta}=b\sec^2\theta$$

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}=\frac{b}{a}\cdot\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}$$

$$y_1=\frac{b}{a}\operatorname{cosec}\theta$$

$$\frac{d^2y}{dx^2}=\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx}$$

$$=\frac{d}{d\theta}\left(\frac{b}{a}\operatorname{cosec}\theta\right)\cdot\frac{1}{\frac{dx}{d\theta}}$$

$$=\frac{b}{a}(-\operatorname{cosec}\theta\cot\theta)\cdot\frac{1}{a\sec\theta\tan\theta}$$

$$= \frac{-b}{a^2}\left[\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}\cdot\cos\theta\cdot\frac{\cos\theta}{\sin\theta}\right]$$

$$= \frac{-b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}$$

$$y_2 = \frac{-b}{a^2}\cot^3\theta$$

Consider

$$\left(\frac{1+y_1^2}{y_2}\right) = \frac{1+\left(\frac{b}{a}\right)^2\operatorname{cosec}^2\theta}{\frac{-b}{a^2}\cot^3\theta} = \frac{-(a^2+b^2\operatorname{cosec}^2\theta)}{b\cot^3\theta}$$

$$\therefore\ \bar{x} = a\sec\theta + \left(\frac{b}{a}\right)\operatorname{cosec}\theta\left(\frac{a^2+b^2\operatorname{cosec}^2\theta}{b\cot^3\theta}\right)$$

$$= a\sec\theta + \frac{b}{a}\cdot\frac{1}{\sin\theta}\left(\frac{a^2+b^2\frac{1}{\sin^2\theta}}{b\cdot\frac{\cos^3\theta}{\sin^3\theta}}\right)$$

$$= a\sec\theta + \frac{1}{a}\left(\frac{a^2\sin^2\theta+b^2}{\cos^3\theta}\right)$$

$$= \frac{a^2\cos^2\theta + a^2\sin^2\theta + b^2}{a\cos^3\theta}$$

$$\bar{x} = \left(\frac{a^2+b^2}{a}\right)\sec^3\theta$$

Similarly,

$$\bar{y} = b\tan\theta - \left[\frac{a^2 + b^2 \operatorname{cosec}^2\theta}{b\cot^3\theta}\right]$$

$$= b\tan\theta - \left[\frac{a^2 + b^2 \dfrac{1}{\sin^2\theta}}{b\dfrac{\cos^3\theta}{\sin^3\theta}}\right]$$

$$= b\frac{\sin\theta}{\cos\theta} - \left[\frac{a^2\sin^2\theta + b^2}{b\dfrac{\cos^3\theta}{\sin\theta}}\right]$$

$$= \frac{b\sin\theta}{\cos\theta} - \frac{\sin\theta}{b\cos^3\theta}\left[a^2\sin^2\theta + b^2\right]$$

$$= \frac{b^2\sin\theta\cos^2\theta - \sin\theta(a^2\sin^2\theta + b^2)}{b\cos^3\theta}$$

$$= \frac{b^2\sin\theta(1-\sin^2\theta) - a^2\sin^3\theta - b^2\sin\theta}{b\cos^3\theta}$$

$$= \frac{b^2\sin\theta - b^2\sin^3\theta - a^2\sin^3\theta - b^2\sin\theta}{b\cos^3\theta}$$

$$= \frac{-(a^2+b^2)}{b}\frac{\sin^3\theta}{\cos^3\theta}$$

$$\therefore\ \bar{y} = -\left(\frac{a^2+b^2}{b}\right)\tan^3\theta$$