UNIT 3– DIFFERENTIAL CALCULUS Evolute

1. Find the equation of the rectangular hyperbola $xy = c^2$.

Soln: Given $xy = c^2$ … (1)

Let $P\left(ct, \frac{c}{t}\right)$ be any point on (1)

$$y = \frac{c^2}{x}$$

$$\therefore \frac{dy}{dx} = -\frac{c^2}{x^2} \qquad \Rightarrow y_1 = -\frac{c^2}{x^2}$$

and $$\frac{d^2y}{dx^2} = \frac{2c^2}{x^3} \qquad \Rightarrow y_2 = \frac{2c^2}{x^3}$$

$\therefore$ At $\left(ct, \frac{c}{t}\right)$, $y_1 = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$

and $y_2 = \frac{2c^2}{c^3t^3} = \frac{2}{ct^3}$

$$\therefore y_1 = -\frac{1}{t^2}, \; y_2 = \frac{2}{ct^3}$$

The centre of curvature $(\bar{x}, \bar{y})$ at P is given by

$$\bar{x} = x - y_1\left(\frac{1 + y_1^2}{y_2}\right)$$

$$= ct - \frac{-\frac{1}{t^2}\left(1+\frac{1}{t^4}\right)}{\frac{2}{ct^3}} = ct + \frac{ct}{2}\left(1+\frac{1}{t^4}\right)$$

$$= \frac{3ct}{2} + \frac{c}{2t^3}$$

$$\bar{x} = \frac{c}{2t^3}\left(3t^4+1\right) \qquad \dots (2)$$

$$\bar{y} = y + \left(\frac{1+{y_1}^2}{y_2}\right)$$

$$= \frac{c}{t} + \frac{1+\frac{1}{t^4}}{\frac{2}{ct^3}} = \frac{c}{t} + \frac{c}{2t}(t^4+1) = \frac{3c}{2t} + \frac{ct^4}{2t}$$

$$= \frac{c}{2t^3}\left(3t^2+t^6\right) \qquad \dots (3)$$

$$\therefore \bar{x} + \bar{y} = \frac{c}{2t^3}\left[3t^4+1+3t^2+t^6\right]$$

$$= \frac{c}{2t^3}\left[1+3t^2+3t^4+t^6\right] = \frac{c}{2t^3}\left(1+t^2\right)^3$$

$$\Rightarrow \quad \bar{x} + \bar{y} = \frac{c}{2}\left(\frac{t^2+1}{t}\right)^3$$

$$\Rightarrow \quad (\bar{x} + \bar{y})^{1/3} = \left(\frac{c}{2}\right)^{1/3}\left(\frac{t^2+1}{t}\right) \qquad \dots (4)$$

Also $\bar{x}-\bar{y}=\frac{c}{2t^3}\left[3t^4+1-3t^2-t^6\right]$

$$=\frac{c}{2t^3}\left[1-3t^2+3t^4-t^6\right]=\frac{c}{2t^3}\left(1-t^2\right)^3$$

$$=\frac{c}{2}\left(\frac{1-t^2}{t}\right)^3$$

$$\Rightarrow \quad (\bar{x}-\bar{y})^{1/3}=\left(\frac{c}{2}\right)^{1/3}\left(\frac{1-t^2}{t}\right) \quad \ldots (5)$$

Eliminating t from (4) and (5), we get the equation of the evolute.

$$(\bar{x}+\bar{y})^{\frac{2}{3}}-(\bar{x}-\bar{y})^{\frac{2}{3}}=\left(\frac{c}{2}\right)^{\frac{2}{3}}\left[\left(\frac{1+t^2}{t}\right)^2-\left(\frac{1-t^2}{t}\right)^2\right]$$

$$=\left(\frac{c}{2}\right)^{2/3}\left[\frac{\left(1+t^2\right)^2-\left(1-t^2\right)^2}{t^2}\right]$$

$$=c^{2/3}\cdot 2^{2-\frac{2}{3}}=c^{2/3}\,2^{4/3}=c^{2/3}\left(2^2\right)^{2/3}=c^{2/3}\left(4\right)^{2/3}$$

$$\Rightarrow \quad (\bar{x}+\bar{y})^{2/3}-(\bar{x}-\bar{y})^{2/3}=\left(4c\right)^{2/3}$$

- Locus of $(\bar{x},\bar{y})$ is $(x+y)^{2/3}-(x-y)^{2/3}=\left(4c\right)^{2/3}$, which is the equation of the evolute.

2. Show that the evolute of the cycloid $x=a(\theta-\sin\theta), y=a(1-\cos\theta)$ is another cycloid.

Soln: Let $'\theta'$ be any point on the cycloid.

Given, $x = a(\theta - \sin\theta)$

$$\therefore \frac{dx}{d\theta} = a(1-\cos\theta) = 2a\sin^2\frac{\theta}{2}$$

$$\therefore \frac{dy}{d\theta} = a\sin\theta = 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2a\sin^2\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{d\theta}\left(\cot\dfrac{\theta}{2}\right)\dfrac{d\theta}{dx}$

$$= -\cos ec^2\frac{\theta}{2}\cdot\frac{1}{2}\cdot\frac{1}{2a\sin^2\frac{\theta}{2}}$$

$$= \frac{-\cos ec^4\frac{\theta}{2}}{4a}$$

$$\therefore y_1 = \cot\frac{\theta}{2},\quad y_2 = -\frac{\cos ec^4\frac{\theta}{2}}{4a}$$

The centre of curvature $(\bar{x}, \bar{y})$ at θ is given by

$$\bar{x} = x - y_1\left(\frac{1+y_1^{\,2}}{y_2}\right)$$

$$= a(\theta - \sin\theta) - \cot\frac{\theta}{2}\frac{\left(1+\cot^2\frac{\theta}{2}\right)}{\left(\frac{-\operatorname{cosec}^4\frac{\theta}{2}}{4a}\right)}$$

$$= a(\theta - \sin\theta) + \frac{4a\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\cdot\frac{\operatorname{cosec}^2\frac{\theta}{2}}{\operatorname{cosec}^4\frac{\theta}{2}}$$

$$= a(\theta - \sin\theta) + 4a\cos\frac{\theta}{2}\cdot\sin\frac{\theta}{2}$$

$$= a(\theta - \sin\theta) + 2a\cdot\sin\theta$$

$$= a\theta + a\sin\theta$$

$$\Rightarrow \quad \bar{x} = a(\theta + \sin\theta) \qquad \ldots (1)$$

$$\bar{y} = y + \left(\frac{1+y_1^{\,2}}{y_2}\right)$$

$$= a(1-\cos\theta) + \frac{1+\cot^2\frac{\theta}{2}}{\left(\frac{-\operatorname{cosec}^4\frac{\theta}{2}}{4a}\right)}$$

$$= a(1-\cos\theta) - 4a\frac{\operatorname{cosec}^2\frac{\theta}{2}}{\operatorname{cosec}^4\frac{\theta}{2}}$$

$$= a(1-\cos\theta) - 4a\sin^2\frac{\theta}{2}$$

$$= a\left(2\sin^2\frac{\theta}{2}\right) - 4a\sin^2\frac{\theta}{2}$$

$$= -2a\sin^2\frac{\theta}{2}$$

$$\Rightarrow \quad \bar{y} = -a(1-\cos\theta) \ldots (2)$$

$\therefore$ The locus of $(\bar{x}, \bar{y})$ is given by the parametric equations
$x = a(\theta + \sin\theta), y = -a(1-\cos\theta),$ which is the another cycloid.