Curvature: Curvature of a curve is a measure of rate of change of bendiness.

Radius of curvature to a curve at a point is denoted by ρ and is the reciprocal of the curvature at that point. Thus $\rho = \frac{1}{k}$

Cartesian form: If $y = f(x)$ **,then**

$$\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$$

If $x = f(y)$ **,then**

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left(\frac{d^2x}{dy^2}\right)}$$

1. Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the curve $x^3 + y^3 = 3axy$.

SOLN**:** The radius of curvature at the given point is

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)}$$

Equation of the given curve is $x^3 + y^3 = 3axy$,

Differentiating the equation of the curve w.r.to x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$$

$$i.e., 3\,(y^2 - ax)\frac{dy}{dx} = 3\,(ay - x^2)$$

$$\therefore \frac{dy}{dx} = \frac{(ay - x^2)}{(y^2 - ax)} \qquad \ldots (1)$$

Again differentiating w.r.to. 'x', we get

$$\frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left(a\,\frac{dy}{dx} - 2x\right) - (ay - x^2)\left(2y\frac{dy}{dx} - a\right)}{(y^2 - ax)^2} \qquad \ldots (2)$$

$$\therefore \left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{\left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)} = -1$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)(-a - 3a) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)(-3a - a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2} = -1$$

$$= \frac{\frac{3a^2}{4}(-4a) - \left(\frac{-3a^2}{4}\right)(-4a)}{\left(\frac{3a^2}{4}\right)^2}$$

$$= \frac{-3a^3 \; - 3a^3}{\left(\frac{9a^4}{16}\right)}$$

$$= \frac{-6a^3 \times 16}{9a^4} = \frac{-32}{3a}$$

$$Hence, the\ radius\ of\ curvature\ at\ point \left(\frac{3a}{2},\frac{3a}{2}\right)\ is$$

$$\rho = \frac{[1 + (-1)^2]^{\frac{3}{2}}}{\left(\frac{-32}{3a}\right)} = \frac{-\left(2^{\frac{3}{2}} \times 3a\right)}{32}$$

$$\rho = \frac{-3\sqrt{2}a}{16}$$

$$i.e., \rho = \frac{3\sqrt{2}a}{16}, since\ \rho\ is\ always\ non-negative.$$

$$2. Find\ the\ radius\ of\ curvature\ at\ the\ point\ \left(\frac{1}{4},\frac{1}{4}\right)\ on\ the\ curve\ \sqrt{x}+\sqrt{y}=1$$

SOLN**:** Equation of the given curve is $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1$.

Differentiating w.r.to x, we get

$$\frac{1}{2}x^{\frac{-1}{2}} + \frac{1}{2}y^{\frac{-1}{2}}\frac{dy}{dx} = 0$$

$$i.e., \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Again differentiating w.r.to x, we get

$$\frac{d^2y}{dx^2} = \frac{-\left[\sqrt{x}\frac{1}{2}y^{\frac{-1}{2}}\frac{dy}{dx} - \sqrt{y}\,\frac{1}{2}x^{\frac{-1}{2}}\right]}{x}$$

$$= -\frac{\left[\frac{\sqrt{x}}{2\sqrt{y}}\frac{dy}{dx} - \frac{\sqrt{y}}{2\sqrt{x}}\right]}{x}$$

$$= -\frac{\left[\frac{\sqrt{x}}{2\sqrt{y}}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right) - \frac{\sqrt{y}}{2\sqrt{x}}\right]}{x}$$

$$= \frac{-\left(\frac{1}{2} + \frac{\sqrt{y}}{2\sqrt{x}}\right)}{x}$$

$$= \frac{-(\sqrt{x}+\sqrt{y})}{2x\ \sqrt{x}}$$

$$Now, \left(\frac{dy}{dx}\right)_{\left(\frac{1}{4},\frac{1}{4}\right)} = \frac{-\left(\frac{1}{\sqrt{4}}\right)}{\left(\frac{1}{\sqrt{4}}\right)} = -1$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{1}{4},\frac{1}{4}\right)} = \frac{\left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{4}}\right)}{2.\frac{1}{4}\frac{1}{\sqrt{4}}} = \frac{\left(\frac{2}{\sqrt{4}}\right)}{\left(\frac{1}{2\sqrt{4}}\right)} = 4$$

$$Hence\ , the\ radius\ of\ curvature\ at\ the\ point\ \left(\frac{1}{4},\frac{1}{4}\right)\ is$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)} at \left(\frac{1}{4},\frac{1}{4}\right) = \frac{[1+(-1)^2]^{\frac{3}{2}}}{4} = \frac{2^{\frac{3}{2}}}{4} = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$$

3. Find the radius of curvature at $x = c$ on the curve $xy = c^2$.

Soln:

Given equation of a curve is $xy = c^2$ ... (1)

Since $x = c, cy = c^2 \Rightarrow y = c$

∴ we have to find the radius of curvature at (c, c) on $xy = c^2$

Differentiating equation (1) w.r.to.*x*, we get

$$x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x} \qquad \text{... (2)}$$

$$\therefore \left(\frac{dy}{dx}\right)_{c,c} = \frac{-c}{c} = -1$$

Differentiating (2) w.r.to. *x*, we get

$$\frac{d^2y}{dx^2} = -\frac{\left[x\frac{dy}{dx} - y.1\right]}{x^2} = -\frac{\left[x.\left(\frac{-y}{x}\right) - y\right]}{x^2}$$

$$= \frac{2y}{x^2}$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{(c,c)} = \frac{2c}{c^2} \;= \frac{2}{c}$$

∴ The radius of curvature at (c,c) is

$$\rho \;= \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)} at\ (c, c)$$

$$= \;\frac{[1 + (-1)^2]^{\frac{3}{2}}}{\left(\frac{2}{c}\right)} = \;\frac{2^{\frac{3}{2}}}{\left(\frac{2}{c}\right)} = 2\sqrt{2} \cdot \frac{c}{2}$$

i.e. $\rho = \; c\sqrt{2}$