UNIT 3– DIFFERENTIAL CALCULUS Evolute

1. Find the equation of the rectangular hyperbola $xy = c^2$.

Soln: Given $xy = c^2$ … (1)

Let $P\left(ct, \frac{c}{t}\right)$ be any point on (1)

$$y = \frac{c^2}{x}$$

$$\therefore \frac{dy}{dx} = -\frac{c^2}{x^2} \qquad \Rightarrow y_1 = -\frac{c^2}{x^2}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{2c^2}{x^3} \qquad \Rightarrow y_2 = \frac{2c^2}{x^3}$$

$$\therefore \text{At } \left(ct, \frac{c}{t}\right), y_1 = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$$

$$\text{and } y_2 = \frac{2c^2}{c^3t^3} = \frac{2}{ct^3}$$

$$\therefore y_1 = -\frac{1}{t^2}, \quad y_2 = \frac{2}{ct^3}$$

The centre of curvature $(\bar{x}, \bar{y})$ at P is given by

$$\bar{x} = x - y_1\left(\frac{1 + {y_1}^2}{y_2}\right)$$

$$= ct - \frac{-\frac{1}{t^2}\left(1 + \frac{1}{t^4}\right)}{\frac{2}{ct^3}} = ct + \frac{ct}{2}\left(1 + \frac{1}{t^4}\right)$$

$$= \frac{3ct}{2} + \frac{c}{2t^3}$$

$$\bar{x} = \frac{c}{2t^3}\left(3t^4 + 1\right) \qquad \ldots (2)$$

$$\bar{y} = y + \left(\frac{1 + y_1^{\,2}}{y_2}\right)$$

$$= \frac{c}{t} + \frac{1 + \frac{1}{t^4}}{\frac{2}{ct^3}} = \frac{c}{t} + \frac{c}{2t}(t^4 + 1) = \frac{3c}{2t} + \frac{ct^4}{2t}$$

$$= \frac{c}{2t^3}\left(3t^2 + t^6\right) \ldots (3)$$

$$\therefore \bar{x} + \bar{y} = \frac{c}{2t^3}\left[3t^4 + 1 + 3t^2 + t^6\right]$$

$$= \frac{c}{2t^3}\left[1 + 3t^2 + 3t^4 + t^6\right] = \frac{c}{2t^3}\left(1 + t^2\right)^3$$

$$\Rightarrow \quad \bar{x} + \bar{y} = \frac{c}{2}\left(\frac{t^2 + 1}{t}\right)^3$$

$$\Rightarrow \quad (\bar{x} + \bar{y})^{1/3} = \left(\frac{c}{2}\right)^{1/3}\left(\frac{t^2 + 1}{t}\right) \ldots (4)$$

Also $\bar{x} - \bar{y} = \frac{c}{2t^3}\left[3t^4 + 1 - 3t^2 - t^6\right]$

$$= \frac{c}{2t^3}\left[1 - 3t^2 + 3t^4 - t^6\right] = \frac{c}{2t^3}\left(1 - t^2\right)^3$$

$$= \frac{c}{2}\left(\frac{1-t^2}{t}\right)^3$$

$$\Rightarrow\ (\bar{x}-\bar{y})^{1/3} = \left(\frac{c}{2}\right)^{1/3}\left(\frac{1-t^2}{t}\right) \ldots (5)$$

Eliminating t from (4) and (5), we get the equation of the evolute.

$$(\bar{x}+\bar{y})^{\frac{2}{3}} - (\bar{x}-\bar{y})^{\frac{2}{3}} = \left(\frac{c}{2}\right)^{\frac{2}{3}}\left[\left(\frac{1+t^2}{t}\right)^2 - \left(\frac{1-t^2}{t}\right)^2\right]$$

$$= \left(\frac{c}{2}\right)^{2/3}\left[\frac{(1+t^2)^2-(1-t^2)^2}{t^2}\right]$$

$$= c^{2/3}\cdot 2^{2-\frac{2}{3}} = c^{2/3}\, 2^{4/3} = c^{2/3}\left(2^2\right)^{2/3} = c^{2/3}(4)^{2/3}$$

$$\Rightarrow\ (\bar{x}+\bar{y})^{2/3} - (\bar{x}-\bar{y})^{2/3} = (4c)^{2/3}$$

$\therefore$ Locus of $(\bar{x},\bar{y})$ is $(x+y)^{2/3}-(x-y)^{2/3}=(4c)^{2/3}$, which is the equation of the evolute.

2. Show that the evolute of the cycloid $x = a(\theta - \sin\theta), y = a(1-\cos\theta)$ is another cycloid.

Soln: Let 'θ' be any point on the cycloid.

Given, $x = a(\theta - \sin\theta)$

$$\therefore \frac{dx}{d\theta} = a(1-\cos\theta) = 2a\sin^2\frac{\theta}{2}$$

$$\therefore \frac{dy}{d\theta} = a\sin\theta = 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2a\sin^2\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)\frac{d\theta}{dx}$$

$$= -\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{1}{2}\cdot\frac{1}{2a\sin^2\frac{\theta}{2}}$$

$$= \frac{-\operatorname{cosec}^4\frac{\theta}{2}}{4a}$$

$$\therefore y_1 = \cot\frac{\theta}{2},\ \ y_2 = -\frac{\operatorname{cosec}^4\frac{\theta}{2}}{4a}$$

The centre of curvature $(\bar{x}, \bar{y})$ at θ is given by

$$\bar{x} = x - y_1\left(\frac{1+y_1^{\,2}}{y_2}\right)$$

$$= a(\theta - \sin\theta) - \cot\frac{\theta}{2}\frac{\left(1+\cot^2\frac{\theta}{2}\right)}{\left(\frac{-\operatorname{cosec}^4\frac{\theta}{2}}{4a}\right)}$$

$$= a(\theta - \sin\theta) + \frac{4a\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \cdot \frac{\operatorname{cosec}^2\frac{\theta}{2}}{\operatorname{cosec}^4\frac{\theta}{2}}$$

$$= a(\theta - \sin\theta) + 4a\cos\frac{\theta}{2} \cdot \sin\frac{\theta}{2}$$

$$= a(\theta - \sin\theta) + 2a \cdot \sin\theta$$

$$= a\theta + a\sin\theta$$

$$\Rightarrow \quad \bar{x} = a(\theta + \sin\theta) \qquad \ldots (1)$$

$$\bar{y} = y + \left(\frac{1 + y_1^2}{y_2}\right)$$

$$= a(1 - \cos\theta) + \frac{1 + \cot^2\frac{\theta}{2}}{\left(\frac{-\operatorname{cosec}^4\frac{\theta}{2}}{4a}\right)}$$

$$= a(1 - \cos\theta) - 4a\frac{\operatorname{cosec}^2\frac{\theta}{2}}{\operatorname{cosec}^4\frac{\theta}{2}}$$

$$= a(1 - \cos\theta) - 4a\sin^2\frac{\theta}{2}$$

$$= a\left(2\sin^2\frac{\theta}{2}\right) - 4a\sin^2\frac{\theta}{2}$$

$$= -2a\sin^2\frac{\theta}{2}$$

$\Rightarrow \quad \bar{y} = -a(1-\cos\theta) \ldots (2)$

$\therefore$ The locus of $(\bar{x}, \bar{y})$ is given by the parametric equations
$x = a(\theta + \sin\theta), y = -a(1-\cos\theta),$ which is the another cycloid.