UNIT 3– DIFFERENTIAL CALCULUS   Centre of Curvature

Centre of Curvature:

The centre of curvature of a curve $y = f(x)$ at a point $P(x, y)$ is denoted by $C(\bar{x}, \bar{y})$ and

$$\bar{x} = x - y_1\left(\frac{1 + y_1^2}{y_2}\right)$$ and

$$\bar{y} = y + \left(\frac{1 + y_1^2}{y_2}\right)$$

1. Find the centre of ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ **at** (a, b).

Soln:

Given curve: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ -----> (1)

Point: $P(a, b)$

Differentiating (1) w.r.t 'x', we get

$$\frac{1}{a^2} 2x + \frac{1}{b^2} 2y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y} \cdot \frac{b^2}{a^2}$$ -----> (2)

$$\therefore \frac{dy}{dx} \text{ at } (a, b) = \frac{-a}{b} \cdot \frac{b^2}{a^2} = \frac{-b}{a}$$

$$y = \frac{-b}{a}$$

From (2),

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$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$= \frac{d}{dx}\left[\frac{-x}{y}\cdot\frac{b^2}{a^2}\right]$$

$$= \frac{-b^2}{a^2}\frac{d}{dx}\left[\frac{x}{y}\right]$$

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left[\frac{y(1)-x\dfrac{dy}{dx}}{y^2}\right]$$

i.e.,

$$\therefore \frac{d^2y}{dx^2} \text{ at } P(a,b) = \frac{-b^2}{a^2}\left[\frac{b-a\left(\dfrac{-b}{a}\right)}{b^2}\right]$$

$$y_2 = \frac{2b}{a^2}$$

Consider

$$\frac{1+y_1^{\,2}}{y_2} = \frac{1+\left(\dfrac{-b}{a}\right)^2}{\dfrac{-2b}{a^2}} = \frac{a^2+b^2}{-2b} = \frac{-\left(a^2+b^2\right)}{+2b}$$

$$\bar{x} = a - \left(\frac{-b}{a}\right)\left(\frac{a^2+b^2}{2b}\right) = \frac{2a^2-a^2-b^2}{2a} = \frac{a^2-b^2}{2a}$$

$$\bar{y} = b - \left(\frac{a^2+b^2}{2b}\right) = \frac{2b^2-a^2-b^2}{2b} = -\left(\frac{a^2-b^2}{2b}\right)$$

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$\therefore$ Centre of curvature $C(\bar{x}, \bar{y})$ at $P(a,b)$ is $\bar{x} = \dfrac{a^2 + b^2}{2a}$, $\bar{y} = \dfrac{b^2 - a^2}{2b}$.

2. Find the centre of curvature of hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point $(a\sec\theta, b\tan\theta)$.

Soln:

Given curve: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ -----> (1)

Point: $P(a\sec\theta, b\tan\theta)$

Here given point is the parametric representation of (1).

$$\therefore x = a\sec\theta, \quad \text{-----> (2)}$$

$$y = b\tan\theta, \quad \text{-----> (3) where } \theta \text{ is the parameter.}$$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta, \quad \frac{dy}{d\theta} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a} \cdot \frac{1}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta}$$

$$y_1 = \frac{b}{a}\csc\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx}$$

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$$= \frac{d}{d\theta}\left(\frac{b}{a}\cos ec\,\theta\right)\cdot\frac{1}{\dfrac{dx}{d\theta}}$$

$$= \frac{b}{a}\left(-\cos ec\,\theta\cot\theta\right)\cdot\frac{1}{a\sec\theta\tan\theta}$$

$$= \frac{-b}{a^2}\left[\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}\cdot\cos\theta\cdot\frac{\cos\theta}{\sin\theta}\right]$$

$$= \frac{-b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}$$

$$y_2 = \frac{-b}{a^2}\cot^3\theta$$

Consider

$$\left(\frac{1+y_1^2}{y_2}\right) = \frac{1+\left(\dfrac{b}{a}\right)^2\cos ec^2\theta}{\dfrac{-b}{a^2}\cot^3\theta} = \frac{-(a^2+b^2\cos ec^2\theta)}{b\cot^3\theta}$$

$$\therefore \quad \bar{x} = a\sec\theta + \left(\frac{b}{a}\right)\cos ec\,\theta\left(\frac{a^2+b^2\cos ec^2\theta}{b\cot^3\theta}\right)$$

$$= a\sec\theta + \frac{b}{a}\cdot\frac{1}{\sin\theta}\left(\frac{a^2+b^2\dfrac{1}{\sin^2\theta}}{b\cdot\dfrac{\cos^3\theta}{\sin^3\theta}}\right)$$

$$= a\sec\theta + \frac{1}{a}\left(\frac{a^2\sin^2\theta+b^2}{\cos^3\theta}\right)$$

$$= \frac{a^2\cos^2\theta+a^2\sin^2\theta+b^2}{a\cos^3\theta}$$

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$$\bar{x} = \left(\frac{a^2 + b^2}{a}\right) \sec^3 \theta$$

Similarly,

$$\bar{y} = b\tan\theta - \left[\frac{a^2 + b^2 \cos ec^2 \theta}{b\cot^3 \theta}\right]$$

$$= b\tan\theta - \left[\frac{a^2 + b^2 \dfrac{1}{\sin^2 \theta}}{b\dfrac{\cos^3 \theta}{\sin^3 \theta}}\right]$$

$$= b\frac{\sin\theta}{\cos\theta} - \left[\frac{a^2 \sin^2 \theta + b^2}{b\dfrac{\cos^3 \theta}{\sin\theta}}\right]$$

$$= \frac{b\sin\theta}{\cos\theta} - \frac{\sin\theta}{b\cos^3 \theta}\left[a^2 \sin^2 \theta + b^2\right]$$

$$= \frac{b^2 \sin\theta\cos^2 \theta - \sin\theta(a^2 \sin^2 \theta + b^2)}{b\cos^3 \theta}$$

$$= \frac{b^2 \sin\theta(1 - \sin^2 \theta) - a^2 \sin^3 \theta - b^2 \sin\theta}{b\cos^3 \theta}$$

$$= \frac{b^2 \sin\theta - b^2 \sin^3 \theta - a^2 \sin^3 \theta - b^2 \sin\theta}{b\cos^3 \theta}$$

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$$= \frac{-(a^2+b^2)}{b} \frac{\sin^3 \theta}{\cos^3 \theta}$$

$$\therefore \quad \bar{y} = -\left(\frac{a^2+b^2}{b}\right) \tan^3 \theta$$