→ Diagonalisation of a Matrix :-

The process of finding a matrix N such that $D = N^T A N$ is called the diagonalisation of a matrix A, where N is the normalized model matrix and D is the diagonal matrix whose diagonal elements are Eigen values of a matrix A.

Note :-

Diagonalisation by orthogonal transformation is possible only for a real symmetric matrix

Steps :-

* steps to find the diagonalisation of matrix :-

Step 1, Find the characteristic equation, Eigen Values and Eigen vectors

Step 2, Eigen vectors should be pair vise orthogonal.

Step 3, Find the normalized Eigen vector
↳ $x = \begin{pmatrix} x_1/l(x) \\ x_2/l(x) \\ x_3/l(x) \end{pmatrix}$ where $l(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}$

Step 4, Form the normalized model matrix using normalized Eigen vector.

Step 5, N should be orthogonal.

Step 6, $D = N^T A N$

Problems :-

1) Reduce the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ to diagonal form by orthogonal trans-formation.

Step:- 1
The characteristic equation is

$$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$$

$D_1 = 1 + 3 + 3 = 7$

$D_2 = (9-1) + (3-0) + (3-0)$
$= 8 + 3 + 3 = 14$

$D_3 = |A|$
$= 1(9-1) - 0 + 0 = 8$

$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

Step:-2 To find the eigen values

$$\begin{array}{c|cccc} 1 & 1 & -7 & 14 & -8 \\ & 0 & 1 & -6 & 8 \\ \hline & 1 & -6 & 8 & \boxed{0} \end{array}$$

$\lambda = 1$

$4 \times 2 = 8$
$-4 - 2 = -6$

$\lambda^2 - 6\lambda + 8 = 0$
$(\lambda - 2)(\lambda - 4) = 0$
$\lambda = 2, 4$

$\therefore$ Eigen values are $1, 2, 4$

Step:- 3 To find the eigen-vectors

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

when $\lambda = 1$,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By cross multiplication rule,

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2 & -1 & 0 & 2 \\ -1 & 2 & 0 & -1 \end{array}$$

$$\frac{x_1}{(4-1)} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{0} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Case :- 2

$\lambda = 2$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By cross multiplication rule,

$$\begin{array}{cccc} x_1 & & x_2 & x_3 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{array}$$

$$\frac{x_1}{0} = \frac{x_2}{(0-1)} = \frac{x_3}{(-1-0)}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

Case :- 3

$\lambda = 4$

$$\begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By cross multiplication rule,

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 0 & & 0 & & -3 & 0 \\ -1 & & -1 & & 0 & -1 \end{array}$$

$$\frac{x_1}{0} = \frac{x_2}{(0-3)} = \frac{x_3}{(3-0)}$$

$$\frac{x_1}{0} = \frac{x_2}{-3} = \frac{x_3}{3}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ Step:- 2!

Now

$$X_1^T \cdot X_2 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}_{1\times 3}\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}_{3\times 1} = 0+0+0=0$$

$$X_2^T \cdot X_3 = \begin{pmatrix} 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = (0+1-1) = 0$$

$$X_3^T \cdot X_1 = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0+0+0 = 0$$

$\therefore$ Eigen vectors are pairwise orthogonal.

$\Rightarrow$ Step 3:- Normalized Eigen vectors,

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad l(x) = \sqrt{1^2+0^2+0^2} = 1$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \qquad l(x) = \sqrt{0^2+(-1)^2+(-1)^2} = \sqrt{2}$$

$$X_2 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$x_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad l(x) = \sqrt{0^2+(-1)^2+(1)^2} = \sqrt{2}$$

$$x_3 = \begin{pmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Step 4 :-

Normalized model matrix

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Step 5 :- N is should be orthogonal (To check)

ie, $NN^T = N^TN = I$

Now,

$$NN^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

N is orthogonal.

Step: 6 :-

$$D = N^T A N$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\therefore D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$