



UNIT 3- DIFFERENTIAL CALCULUS

ENVELOPE

Definition

Let $f(x, y, \alpha) = 0$ be a single parameter family of curves, where α is the parameter. The envelope of this family of curves is a curve which touches every member of the family.

1. Find the envelope of the family of straight lines $y = mx + \frac{1}{m}$.

Soln: $y = mx + \frac{1}{m} \Rightarrow my = m^2x + 1$

$$\Rightarrow xm^2 - ym + 1 = 0, \text{ which is a quadratic in the parameter } m.$$

$$\text{Here } A = x, B = -y, C = 1$$

$$\therefore \text{ The envelope is } B^2 - 4AC = 0$$

$$\Rightarrow y^2 - 4x = 0$$

$$\Rightarrow y^2 = 4x.$$

2. Find the envelope of the family of lines

$$y = mx \pm \sqrt{a^2m^2 - b^2}, \text{ where } m \text{ is the parameter.}$$

Soln: Given the family is $y = mx \pm \sqrt{a^2m^2 - b^2}$, m is the parameter.

$$\Rightarrow y - mx = \pm \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow (y - mx)^2 = a^2m^2 - b^2$$

$$\Rightarrow y^2 - 2mxy + m^2x^2 = a^2m^2 - b^2$$

$$\Rightarrow m^2(x^2 - a^2) - 2mxy + (y^2 + b^2) = 0$$

This is a quadratic in m

$$\text{Here } A = x^2 - a^2, B = -2xy, C = y^2 + b^2$$



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∴ The envelope is $B^2 - 4AC = 0$

$$\Rightarrow 4x^2y^2 - 4(x^2 - a^2)(y^2 + b^2) = 0$$

$$\Rightarrow x^2y^2 - (x^2y^2 + b^2x^2 - a^2y^2 - a^2b^2) = 0 \quad [\div \text{by } 4]$$

$$\Rightarrow b^2x^2 - a^2y^2 - a^2b^2 = 0$$

$$\Rightarrow b^2x^2 - a^2y^2 = a^2b^2$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

3. Find the envelope of the straight lines represented by

$$x \cos \alpha + y \sin \alpha = a \sec \alpha, \text{ where } \alpha \text{ is a parameter.}$$

Soln: Given $x \cos \alpha + y \sin \alpha = a \sec \alpha$

Dividing by $\cos \alpha$, $x + y \tan \alpha = a \sec^2 \alpha$

$$= a(1 + \tan^2 \alpha)$$

$$\Rightarrow a \tan^2 \alpha - y \tan \alpha + (a - x) = 0,$$

which is a quadratic in $\tan \alpha$

Here $A = a$, $B = -y$, $C = a - x$

∴ The envelope is $B^2 - 4AC = 0 \Rightarrow y^2 - 4a(a - x) = 0$, which is the envelope.

is the envelope and it is an ellipse.

4. Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters a and b are related by

the equation $a^n + b^n = c^n$, c being a constant.



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Soln: Given $\frac{x}{a} + \frac{y}{b} = 1$... (1)

$$a^n + b^n = c^n \quad \dots (2)$$

Differentiating (1) and (2) with respect to a , treating b as a function of a , we get

$$(1) \Rightarrow \frac{x}{a} + \frac{y}{b} = 1 \text{ Diff. w.r.to } a$$

$$-\frac{x}{a^2} - \frac{y}{b^2} \frac{db}{da} = 0 \Rightarrow \frac{db}{da} = -\frac{b^2 x}{a^2 y} \quad \dots (3)$$

and (2) $\Rightarrow a^n + b^n = c^n$ Diff. w.r.to a

$$n a^{n-1} + n b^{n-1} \cdot \frac{db}{da} = 0 \Rightarrow \frac{db}{da} = -\frac{a^{n-1}}{b^{n-1}} \quad \dots (4)$$

From (3) and (4); $\frac{-b^2 x}{a^2 y} = -\frac{a^{n-1}}{b^{n-1}}$

$$\Rightarrow \frac{x}{a^{n+1}} = \frac{y}{b^{n+1}}$$

$$\Rightarrow \frac{\frac{x}{a^n}}{a} = \frac{\frac{y}{b^n}}{b} = \frac{\frac{x}{a} + \frac{y}{b}}{a^n + b^n} = \frac{1}{c^n} \text{ [Using (2)]}$$

$$\therefore \frac{x}{a^{n+1}} = \frac{1}{c^n}$$

$$\Rightarrow a = (c^n x)^{\frac{1}{n+1}}$$

Similarly $b = (c^n y)^{\frac{1}{n+1}}$

Substitute in (1)



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$$\frac{x}{c^{n/n+1} \cdot x^{1/n+1}} + \frac{y}{c^{n/n+1} \cdot y^{1/n+1}} = 1$$

$$\Rightarrow x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}},$$

which is the required envelope.