Circle of Curvature:

The circle of curvature of a curve $y = f(x)$ at a point $P(x, y)$ with the centre of curvature, $C(\bar{x}, \bar{y})$ and radius curvature ρ is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

1. Find the circle of curvature of the curve $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Soln :

Given $x^3 + y^3 = 3axy$ ... (1)

Differentiating (1) w.r.to. 'x', we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3\,a\left[x\frac{dy}{dx} + y\right]$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \qquad \text{... (2)}$$

$$y_1 = \left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)}$$

$$= \frac{a\frac{3a}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a\frac{3a}{2}} = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a\frac{3a^2}{2}}$$

$$= -1$$

$$y_1 = -1$$

From (2), $$\frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left(a\frac{dy}{dx} - 2x\right) - (ay - x^2)\left(2y\frac{dy}{dx} - a\right)}{(y^2 - ax)^2}$$

$$y_2 = \left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)}$$

$$= \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)\left(-a - \frac{6a}{2}\right) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)\left(-\frac{6a}{2} - a\right)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{(-4a) - 4a}{\frac{9a^2}{4} - \frac{3a^2}{2}}$$

$$= \frac{-8a}{\frac{3a^2}{4}} = \frac{-32}{3a}$$

$$y_2 = \frac{-32}{3a}$$

∴ Radius of curvature

$$\therefore \qquad \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{[1 + (-1)^2]^{3/2}}{\frac{-32}{3a}}$$

$$= \frac{(2)^{3/2} \times 3\,a}{-32}$$

$$= \frac{3\sqrt{2} \times a}{16}$$

∴ The centre of curvature

$$\overline{x} = x - y_1\left(\frac{1 + y_1^{\,2}}{y_2}\right)$$

$$= \frac{3a}{2} - \frac{(-1)(2)}{-32} 3a = \frac{21a}{16}$$

$$\bar{y} = y + \left(\frac{1 + y_1^{\,2}}{y_2} \right)$$

$$= \frac{3a}{2} + \frac{(2)}{-32} \cdot 3a = \frac{21a}{16}$$

∴ The required circle of curvature is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

$$\left(x - \frac{21a}{16}\right)^2 + \left(y - \frac{21a}{16}\right)^2 = \left(\frac{3\sqrt{2} \times a}{16}\right)^2$$

$$i.e., \ \ (x^2 + y^2) - \frac{21a}{8}(x + y) + \frac{432a^2}{128} = 0$$

2. Find the circle of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$.

Soln:

Given curve: $\sqrt{x} + \sqrt{y} = \sqrt{a}$ ----> (1)

Point: $P\left(\frac{a}{4}, \frac{a}{4}\right)$

Centre of curvature $C(\bar{x}, \bar{y})$,

$$\bar{x} = x - y_1 \left(\frac{1 + y_1^{\,2}}{y_2} \right)$$

$$\bar{y} = y + \left(\frac{1 + y_1^{\,2}}{y_2} \right)$$

From (1),

$\sqrt{x}+\sqrt{y}=\sqrt{a} \Rightarrow$ Differentiate w.r.to. x

$$\frac{d}{dx}\left(\sqrt{x}+\sqrt{y}\right)=\frac{d}{dx}\left(\sqrt{a}\right)$$

$$\frac{1}{2}x^{-1/2}+\frac{1}{2}y^{-1/2}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x^{-1/2}}{y^{-1/2}}=-\frac{y^{1/2}}{x^{1/2}}$$

at $P\left(\frac{a}{4},\frac{a}{4}\right)$, $\frac{dy}{dx}=-1$

$y_1=-1$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$=\frac{d}{dx}\left(\frac{-y^{1/2}}{x^{1/2}}\right)$$

$$=-\left[\frac{x^{1/2}\left(\frac{1}{2}y^{-1/2}\right)\left(\frac{dy}{dx}\right)-y^{-1/2}\left(\frac{1}{2}x^{-1/2}\right)}{x}\right]$$

$$=-\frac{1}{2}\left[\frac{x^{1/2}\,y^{-1/2}\frac{dy}{dx}-y^{-1/2}\,x^{-1/2}}{x}\right]$$

At $P\left(\frac{a}{4},\frac{a}{4}\right)$, $\frac{d^2y}{dx^2}=\frac{-1}{2}\left[\frac{(1)(-1)-(1)}{\frac{a}{4}}\right]$

$$= \frac{-1}{2}\left[\frac{-2}{\frac{a}{4}}\right]$$

$$y_2 = \frac{4}{a}$$

Consider $\frac{1+y_1^2}{y_2} = \frac{1+(-1)^2}{\frac{4}{a}} = \frac{2}{\frac{4}{a}} = \frac{a}{2}$

$$\bar{x} = \frac{a}{4} - (-1)\left(\frac{a}{2}\right) = \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}$$

$$\bar{y} = \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}$$

We know that

$$\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$$

$$\therefore \text{ At } P\left(\frac{a}{4}, \frac{a}{4}\right), \ \rho = \frac{\left[1+(-1)^2\right]^{3/2}}{\frac{4}{a}} = (2)^{3/2} \times \frac{a}{4}$$

$$= 2\sqrt{2} \times \frac{a}{4}$$

$$= \frac{a\sqrt{2}}{2}$$

$$\therefore \ \rho = \frac{a\sqrt{2}}{2}$$

From (2), we have

$$\left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\left(\frac{a}{\sqrt{2}}\right)^2,$$

3. Find the equation of the circle of curvature at (c, c) on $xy=c^2$.

Soln:

Given curve: $xy=c^2$ ----> (1)

Point: $P(c, c)$

Centre of curvature $C(\bar{x}, \bar{y})$,

$$\bar{x}=x-y_1\left(\frac{1+y_1^2}{y_2}\right)$$

$$\bar{y}=y+\left(\frac{1+y_1^2}{y_2}\right) \text{ where } y_1=\frac{dy}{dx};\ y_2=\frac{d^2y}{dx^2}.$$

From (1),

$$xy=c^2$$

$$\frac{d}{dx}(xy)=\frac{d}{dx}(c^2)$$

$$x\frac{dy}{dx}+y(1)=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-y}{x}$$

At $P(c, c)$, $\frac{dy}{dx}=\frac{-c}{c}=-1$

$$y_1=-1$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{-y}{x}\right)$$

$$= -\left[\frac{x\dfrac{dy}{dx} - y(1)}{x^2}\right]$$

$\therefore$ At $P(c, c)$, $\dfrac{d^2y}{dx^2} = -\left[\dfrac{c(-1)-(c)}{c^2}\right]$

$$= \frac{2c}{c^2}$$

$\therefore$

$$y_2 = \frac{2}{c}$$

Consider $\dfrac{1+y_1^2}{y_2} = \dfrac{1+(-1)^2}{\frac{2}{c}} = \dfrac{2}{\frac{2}{c}} = c$

$$\bar{x} = c - (-1)\,(c) = 2c$$

$$\bar{y} = c + c = 2c$$

We know that

$$\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$$

$\therefore$ at $P(c, c)$, $\rho = \dfrac{\left(1+(-1)^2\right)^{3/2}}{\frac{2}{c}} = (2)^{3/2} \times \dfrac{c}{2}$

$$= 2\sqrt{2} \times \frac{c}{2}$$

$= c\sqrt{2}$

$\therefore\ \rho = c\sqrt{2}$

$\therefore$ (2) becomes

$(x-2c)^2 + (y-2c)^2 = \left(c\sqrt{2}\right)^2$.