Q. Find the eigen values and eigen vector of the matrix :-

ii) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

Step :- 1

The characteristic equation is

$$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$$

$$D_1 = 2+3+2 = 7$$

$$D_2 = (6-0) + (4-1) + (6-0)$$
$$= 6+3+6 = 15$$

$$D_3 = 2(6-0) - 0 + 1(0-3)$$
$$= 2(6) - 3 = 12-3 = 9$$

$$\therefore \lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$$

Step :- 2 To find the eigen values

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 15 & -9 \\ & 0 & 1 & -6 & 9 \\ \hline & 1 & -6 & 9 & \boxed{0} \end{array}$$

$$\lambda = 1$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$3 \times 3 = 9$

$$(\lambda-3)(\lambda-3) = 0$$

$-3 -3 = -6$

$$\lambda = 3, 3$$

The eigen value is 1, 3, 3

Step :- 3 To find the eigen vectors

$$(A - \lambda I)\, x = 0$$

$$\begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case :- 1

when $\lambda = 1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By cross multiplication rule,

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ 0 & & 1 & & 1 & & 0 \\ 2 & & 0 & & 0 & & 2 \end{matrix}$$

$$\frac{x_1}{(0-2)} = \frac{x_2}{(0-0)} = \frac{x_3}{(2-0)}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Case :- 2

when $\lambda = 3$,

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 + 0x_2 + x_3 = 0$$

$$-x_1 + x_3 = 0$$

$$\cancel{+}x_1 = \cancel{+}x_3$$

$$\frac{x_1}{1} = \frac{x_3}{1}$$

$\therefore X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Conclusion:-

Eigen values	Eigen vectors
1	$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$
3	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
3	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Q) Find the eigen values and eigen vectors of the matrix.

8) $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

Step 1:-

The characteristic equation is

$$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$$

$D_1 = 1 + 5 + 1 = 7$

$D_2 = (5-1) + (1-9) + (5-1)$

$= 4 - 8 + 4 = -4 + 4$

$= 0$

$D_3 = 1(5-1) - 1(1-3) + 3(1-15)$

$= 4 - 1(-2) + 3(-14)$

$= 4 + 2 - 42$

$= -36$

$\therefore \lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$

$$\lambda^3 - 7\lambda^2 + \lambda + 36 = 0$$

Step :- 2 To find the eigen values.

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\lambda = -2$$

$$\lambda^2 - 9\lambda + 18 = 0$$

$$(\lambda - 6)(\lambda - 3)$$

$$\lambda = 6, 3$$

$6 \times 3 = 18$
$-6 - 3 = -9$

$\therefore$ The eigen values are $-2, 3, 6$

Step :- 3

To find the eigen vectors

$$(A - \lambda I)x = 0$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case :- 1

$\lambda = -2$

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 3 & 3 & 1 \\ 7 & 1 & 1 & 7 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}$$

$$= \frac{x_1}{(1-21)} = \frac{x_2}{(3-3)} = \frac{x_3}{(20-1)}$$

$$= \frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$

$$= \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Case :- 2

$\lambda = 3$

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & 3 & & -2 & & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{(1-6)} = \frac{x_2}{(3+2)} = \frac{x_3}{(-4-1)}$$

$$= \frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$$= \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$\therefore \quad X_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

Case : 3

$\lambda = 6$

$$\begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 3 & -5 & 1 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\frac{x_1}{\begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}} = \frac{x_2}{\begin{pmatrix} 3 & -5 \\ 1 & 1 \end{pmatrix}} = \frac{x_3}{\begin{pmatrix} -5 & 1 \\ 1 & -1 \end{pmatrix}}$$

$$\frac{x_1}{(1+3)} = \frac{x_2}{(3+5)} = \frac{x_3}{(5-1)}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Conclusion :-

Eigen values	Eigen vectors
-2	$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$
3	$\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$
6	$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Q) Find the eigen values and eigen vectors of the matrix.

9. $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Step 1:-

The characteristic equation is $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

$D_1 = 2+2+2 = 6$

$D_2 = (4-0) + (4-0) + (4-0)$

$= 4+4+4 = 12$

$D_3 = 2(4-0) - 1(0) + 0(0)$

$= 2(4) = 8$

$\therefore \lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

Step:- 2 To find the eigen values

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & 0 & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$\lambda = 2$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda-2)(\lambda-2) = 0$

$\lambda = 2, 2$

$\therefore$ The eigen values are 2, 2, 2

Step:-3 To find the eigen vectors

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case:- 1

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ 1 & & 0 & & 0 & & 1 \\ 0 & & 1 & & 0 & & 0 \end{matrix}$$

$$\frac{x_1}{(1-0)} = \frac{x_2}{(0-0)} = \frac{x_3}{(0-0)}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Conclusion:-

Eigen values	Eigen vectors
2	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
2	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
2	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$