

## UNIT 2 – ORTHOGONAL TRANSFORMATION OF A REAL SYMMETRIC MATRIX

Canonical form

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### Canonical form

Reduce the quadratic form  $2x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 = 0$  to canonical form by orthogonal reduction .Find rank, index, signature and nature

#### **Step 1:**

The matrix form is

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

#### **Step 2:**

Characteristic equation ,Eigen values,Eigen vectors

$C_1$  =Sum of leading diadonal elements

$$=2+2+1 =5$$

$C_2$ = Sum of minors of leading diagonal elements

$$=4$$

$C_3=|A|$

$$= \begin{vmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 0$$

The characteristic equation is

$$\lambda^3 - 5\lambda^2 + 4\lambda = 0$$

The eigen values are 0,1,4

The eigen vectors are  $(A - \lambda I)X=0$

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$$\left[ \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

### CASE (i)

When  $\lambda = 0$

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The cofactor of first row elements are  $\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$  ie  $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

The Eigen vector when  $\lambda = 0$  is  $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

### CASE (ii)

When  $\lambda = 1$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The cofactor of third row elements are  $\begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix}$  ie  $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

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The Eigen vector when  $\lambda = 1$  is  $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

### **CASE (iii)**

When  $\lambda = 4$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The cofactor of first row elements are  $\begin{pmatrix} 6 \\ 6 \\ 0 \end{pmatrix}$  ie  $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

The Eigen vector when  $\lambda = 4$  is  $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

### **STEP 3:**

To check pair wise orthogonality

$$X_1^T X_2 = (2 \quad -2 \quad 0) \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0$$

$$X_2^T X_3 = (0 \quad 0 \quad -1) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$X_3^T X_1 = (1 \quad 1 \quad 0) \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = 0$$

### **STEP 4:**

To find normalized vector

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| Eigen vector                                 | $l(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}$ | Normalized vector = $\begin{pmatrix} x_1/l(x_1) \\ x_2/l(x_2) \\ x_3/l(x_3) \end{pmatrix}$ |
|--|---------------------------------------|--|
| $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ | $\sqrt{1 + 1 + 0} = \sqrt{2}$         | $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$                             |
| $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ | $\sqrt{0 + 0 + 1} = \sqrt{1}$         | $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$   |
| $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$  | $\sqrt{1 + 1 + 0} = \sqrt{2}$         | $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$                              |

### **STEP 5:**

Normalized modal matrix

$$N = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \end{bmatrix}$$

$$N^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{bmatrix}$$

### **STEP 6:**

$$NN^T = N^T N = I$$

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$$\begin{aligned}
 N^T N &= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= I
 \end{aligned}$$

### **STEP 7:**

To find diagonalize matrix

$$N^T A N = D$$

$$N^T A = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 4/\sqrt{2} & 4/\sqrt{2} & 0 \end{pmatrix}$$

$$N^T A N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 4/\sqrt{2} & 4/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= D$$

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### **Step 8:**

$$Y^T D Y = 0$$

$$0y_1^2 + y_2^2 + 4y_3^2 = 0$$

The index  $p=2$

Rank  $r=2$

Signature  $s=2p-r=2$

The nature is semi positive