



## UNIT 3- DIFFERENTIAL CALCULUS

## Evolute

1. Find the equation of the rectangular hyperbola  $xy = c^2$ .

**Soln:** Given  $xy = c^2$  ... (1)

Let  $P\left(ct, \frac{c}{t}\right)$  be any point on (1)

$$y = \frac{c^2}{x}$$

$$\therefore \frac{dy}{dx} = -\frac{c^2}{x^2} \Rightarrow y_1 = -\frac{c^2}{x^2}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{2c^2}{x^3} \Rightarrow y_2 = \frac{2c^2}{x^3}$$

$$\therefore \text{At } \left(ct, \frac{c}{t}\right), y_1 = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$$

$$\text{and } y_2 = \frac{2c^2}{c^3t^3} = \frac{2}{ct^3}$$

$$\therefore y_1 = -\frac{1}{t^2}, y_2 = \frac{2}{ct^3}$$

The centre of curvature  $(\bar{x}, \bar{y})$  at P is given by

$$\begin{aligned} \bar{x} &= x - y_1 \left( \frac{1 + y_1^2}{y_2} \right) \\ &= ct - \frac{-\frac{1}{t^2} \left( 1 + \frac{1}{t^4} \right)}{\frac{2}{ct^3}} = ct + \frac{ct}{2} \left( 1 + \frac{1}{t^4} \right) \end{aligned}$$



## UNIT 3- DIFFERENTIAL CALCULUS

Evolute

$$= \frac{3ct}{2} + \frac{c}{2t^3}$$

$$\bar{x} = \frac{c}{2t^3}(3t^4 + 1) \quad \dots (2)$$

$$\bar{y} = y + \left( \frac{1 + y_1^2}{y_2} \right)$$

$$= \frac{c}{t} + \frac{1 + \frac{1}{t^4}}{\frac{ct^3}{2}} = \frac{c}{t} + \frac{c}{2t}(t^4 + 1) = \frac{3c}{2t} + \frac{ct^4}{2t}$$

$$= \frac{c}{2t^3}(3t^2 + t^6) \dots (3)$$

$$\therefore \bar{x} + \bar{y} = \frac{c}{2t^3}[3t^4 + 1 + 3t^2 + t^6]$$

$$= \frac{c}{2t^3}[1 + 3t^2 + 3t^4 + t^6] = \frac{c}{2t^3}(1 + t^2)^3$$

$$\Rightarrow \bar{x} + \bar{y} = \frac{c}{2} \left( \frac{t^2 + 1}{t} \right)^3$$

$$\Rightarrow (\bar{x} + \bar{y})^{1/3} = \left( \frac{c}{2} \right)^{1/3} \left( \frac{t^2 + 1}{t} \right) \dots (4)$$

$$\text{Also } \bar{x} - \bar{y} = \frac{c}{2t^3}[3t^4 + 1 - 3t^2 - t^6]$$

$$= \frac{c}{2t^3}[1 - 3t^2 + 3t^4 - t^6] = \frac{c}{2t^3}(1 - t^2)^3$$



## UNIT 3- DIFFERENTIAL CALCULUS

## Evolute

$$= \frac{c}{2} \left( \frac{1-t^2}{t} \right)^3$$

$$\Rightarrow (\bar{x} - \bar{y})^{1/3} = \left( \frac{c}{2} \right)^{1/3} \left( \frac{1-t^2}{t} \right) \dots (5)$$

Eliminating  $t$  from (4) and (5), we get the equation of the evolute.

$$(\bar{x} + \bar{y})^{2/3} - (\bar{x} - \bar{y})^{2/3} = \left( \frac{c}{2} \right)^{2/3} \left[ \left( \frac{1+t^2}{t} \right)^2 - \left( \frac{1-t^2}{t} \right)^2 \right]$$

$$= \left( \frac{c}{2} \right)^{2/3} \left[ \frac{(1+t^2)^2 - (1-t^2)^2}{t^2} \right]$$

$$= c^{2/3} \cdot 2^{2-\frac{2}{3}} = c^{2/3} 2^{4/3} = c^{2/3} (2^2)^{2/3} = c^{2/3} (4)^{2/3}$$

$$\Rightarrow (\bar{x} + \bar{y})^{2/3} - (\bar{x} - \bar{y})^{2/3} = (4c)^{2/3}$$

$\therefore$  Locus of  $(\bar{x}, \bar{y})$  is  $(x+y)^{2/3} - (x-y)^{2/3} = (4c)^{2/3}$ , which is the equation of the evolute.

2. Show that the evolute of the cycloid  $x = a(\theta - \sin \theta)$ ,  $y = a(1 - \cos \theta)$  is another cycloid.

**Soln:** Let ' $\theta$ ' be any point on the cycloid.

Given,  $x = a(\theta - \sin \theta)$

$$\therefore \frac{dx}{d\theta} = a(1 - \cos \theta) = 2a \sin^2 \frac{\theta}{2}$$

$$\therefore \frac{dy}{d\theta} = a \sin \theta = 2a \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$



## UNIT 3- DIFFERENTIAL CALCULUS

Evolute

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2a \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2a \sin^2 \frac{\theta}{2}}$$

$$= \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = \cot \frac{\theta}{2}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{d\theta} \left( \cot \frac{\theta}{2} \right) \frac{d\theta}{dx}$$
$$= -\operatorname{cosec}^2 \frac{\theta}{2} \cdot \frac{1}{2} \cdot \frac{1}{2a \sin^2 \frac{\theta}{2}}$$

$$= \frac{-\operatorname{cosec}^4 \frac{\theta}{2}}{4a}$$

$$\therefore y_1 = \cot \frac{\theta}{2}, \quad y_2 = -\frac{\operatorname{cosec}^4 \frac{\theta}{2}}{4a}$$

The centre of curvature  $(\bar{x}, \bar{y})$  at  $\theta$  is given by

$$\bar{x} = x - y_1 \left( \frac{1 + y_1^2}{y_2} \right)$$

$$= a(\theta - \sin \theta) - \cot \frac{\theta}{2} \left( \frac{1 + \cot^2 \frac{\theta}{2}}{\frac{-\operatorname{cosec}^4 \frac{\theta}{2}}{4a}} \right)$$



## UNIT 3- DIFFERENTIAL CALCULUS

Evolute

$$\begin{aligned} &= a(\theta - \sin \theta) + \frac{4a \cos \frac{\theta}{2} \operatorname{cosec}^2 \frac{\theta}{2}}{\sin \frac{\theta}{2} \operatorname{cosec}^4 \frac{\theta}{2}} \\ &= a(\theta - \sin \theta) + 4a \cos \frac{\theta}{2} \cdot \sin \frac{\theta}{2} \\ &= a(\theta - \sin \theta) + 2a \cdot \sin \theta \\ &= a\theta + a \sin \theta \\ \Rightarrow \quad \bar{x} &= a(\theta + \sin \theta) \quad \dots (1) \end{aligned}$$

$$\begin{aligned} \bar{y} &= y + \left( \frac{1 + y_1^2}{y_2} \right) \\ &= a(1 - \cos \theta) + \frac{1 + \cot^2 \frac{\theta}{2}}{\left( \frac{-\operatorname{cosec}^4 \frac{\theta}{2}}{4a} \right)} \\ &= a(1 - \cos \theta) - 4a \frac{\operatorname{cosec}^2 \frac{\theta}{2}}{\operatorname{cosec}^4 \frac{\theta}{2}} \\ &= a(1 - \cos \theta) - 4a \sin^2 \frac{\theta}{2} \\ &= a \left( 2 \sin^2 \frac{\theta}{2} \right) - 4a \sin^2 \frac{\theta}{2} \\ &= -2a \sin^2 \frac{\theta}{2} \end{aligned}$$



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UNIT 3- DIFFERENTIAL CALCULUS

Evolute

$$\Rightarrow \bar{y} = -a(1 - \cos \theta) \dots (2)$$

$\therefore$  The locus of  $(\bar{x}, \bar{y})$  is given by the parametric equations

$x = a(\theta + \sin \theta)$ ,  $y = -a(1 - \cos \theta)$ , which is the another cycloid.