Diagonalization of a real symmetric matrix

Method to diagonalyse the symmetric matrix by orthogonal transformation

STEP 1:Find characteristic equation ,Eigen values,eigen vectors of the given matrix A.

STEP 2:Eigen vector should be pair wise orthogonal

$X_1^TX_2=0$; $X_2^TX_3=0$; $X_3^TX_1=0$

STEP 3: Normalize each Eigen vector, if $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ the normalized vector $=\begin{pmatrix} x_1/l(x_1) \\ x_2/l(x_2) \\ x_3/l(x_3) \end{pmatrix}$ where

$l(x)=\sqrt{x_1^2 + x_2^2 + x_3^2}$

STEP 4: Form the normalized model matrix N, using normalized Eigen vectors

STEP 5: The normalized modal matrix N should be orthogonal matrix

$NN^T = N^TN = I$

STEP 6: Find the $N^TAN = D$,Where D is diagonal matrix.

Problem

1.Reduce the matrix to diagonalise $A=\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

To find characteristic equation

D_1 =Sum of leading diadonal elements

=8+7+3 =18

D_2= Sum of minors of leading diagonal elements

=45

$D_3 = |A|$

$$= \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

= 0

The characteristic equation is

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

The eigen values are 0,3,15

The eigen vectors are $(A - \lambda I)X$=0

$$\left[\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

CASE (i)

When $\lambda = 0$

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The cofactor of first row elements are $\begin{pmatrix} 5 \\ 10 \\ 10 \end{pmatrix}$ ie $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

The Eigen vector when $\lambda = 0$ is $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

CASE (ii)

When $\lambda = 3$

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The cofactor of first row elements are $\begin{pmatrix} -16 \\ -8 \\ 16 \end{pmatrix}$ ie $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

The Eigen vector when $\lambda = 3$ is $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

CASE (iii)

When $\lambda = 15$

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The cofactor of first row elements are $\begin{pmatrix} 80 \\ -80 \\ 40 \end{pmatrix}$ ie $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

The Eigen vector when $\lambda = 15$ is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

STEP 2:

To check pair wise orthgonality

$$X_1^T X_2 = (1 \quad 2 \quad 2)\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 0$$

$$X_2^T X_3 = (2 \quad 1 \quad -2)\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 0$$

$$X_3^T X_1 = (2 \quad -2 \quad 1)\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 0$$

STEP 3:

To find normalized vector

Eigen vector	l(x)=$\sqrt{x_1^2 + x_2^2 + x_3^2}$	Normalized vector $=\begin{pmatrix} x_1/l(x_1) \\ x_2/l(x_2) \\ x_3/l(x_3) \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$	$\sqrt{1+4+4}$ =3	$\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$
$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$	$\sqrt{1+4+4}$ =3	$\begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}$
$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$	$\sqrt{1+4+4}$ =3	$\begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$

STEP 4:

Normalized modal matrix

$$N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$N^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

STEP 5:

$NN^T = N^TN = I$

$$N^TN = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}\begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= I$

STEP 6:

To find diagonalyze matrix

$N^TAN = D$

$$N^TA = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 \\ 6 & 3 & -6 \\ 30 & 30 & 15 \end{pmatrix}$$

$$N^TAN = \frac{1}{9}\begin{pmatrix} 0 & 0 & 0 \\ 6 & 3 & -6 \\ 30 & 30 & 15 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

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