



## UNIT 3- DIFFERENTIAL CALCULUS

## Circle of Curvature

### Circle of Curvature:

The circle of curvature of a curve  $y = f(x)$  at a point  $P(x, y)$  with the centre of curvature,  $C(\bar{x}, \bar{y})$  and radius curvature  $\rho$  is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

1. Find the circle of curvature of the curve  $x^3 + y^3 = 3axy$  at the

point  $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Soln :

$$\text{Given } x^3 + y^3 = 3axy \quad \dots (1)$$

Differentiating (1) w.r.to. 'x', we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a \left[ x \frac{dy}{dx} + y \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \quad \dots (2)$$

$$y_1 = \left( \frac{dy}{dx} \right)_{\left( \frac{3a}{2}, \frac{3a}{2} \right)}$$

$$= \frac{a \frac{3a}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a \frac{3a}{2}} = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a \frac{3a}{2}}$$

$$= -1$$

$$y_1 = -1$$

$$\text{From (2), } \frac{d^2y}{dx^2} = \frac{(y^2 - ax) \left( a \frac{dy}{dx} - 2x \right) - (ay - x^2) \left( 2y \frac{dy}{dx} - a \right)}{(y^2 - ax)^2}$$



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$$\begin{aligned}y_2 &= \left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} \\&= \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)\left(-a - \frac{6a}{2}\right) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)\left(-\frac{6a}{2} - a\right)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2} \\&= \frac{(-4a) - 4a}{\frac{9a^2}{4} - \frac{3a^2}{2}} \\&= \frac{-8a}{\frac{3a^2}{4}} = \frac{-32}{3a} \\y_2 &= \frac{-32}{3a}\end{aligned}$$

∴ Radius of curvature

$$\begin{aligned}\therefore \rho &= \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{[1 + (-1)^2]^{3/2}}{\frac{-32}{3a}} \\&= \frac{(2)^{3/2} \times 3a}{-32} \\&= \frac{3\sqrt{2} \times a}{16}\end{aligned}$$

∴ The centre of curvature

$$\bar{x} = x - y_1 \left(\frac{1 + y_1^2}{y_2}\right)$$



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$$= \frac{3a}{2} - \frac{(-1)(2)}{-32} \cdot 3a = \frac{21a}{16}$$

$$\bar{y} = y + \left( \frac{1 + y_1^2}{y_2} \right)$$

$$= \frac{3a}{2} + \frac{(2)}{-32} \cdot 3a = \frac{21a}{16}$$

∴ The required circle of curvature is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

$$\left( x - \frac{21a}{16} \right)^2 + \left( y - \frac{21a}{16} \right)^2 = \left( \frac{3\sqrt{2} \times a}{16} \right)^2$$

$$i. e., (x^2 + y^2) - \frac{21a}{8}(x + y) + \frac{432a^2}{128} = 0$$

**2. Find the circle of curvature of the curve  $\sqrt{x} + \sqrt{y} = \sqrt{a}$  at the point  $\left( \frac{a}{4}, \frac{a}{4} \right)$ .**

**Soln:**

$$\text{Given curve: } \sqrt{x} + \sqrt{y} = \sqrt{a} \text{ ----> (1)}$$

$$\text{Point: } P\left(\frac{a}{4}, \frac{a}{4}\right)$$

Centre of curvature  $C(\bar{x}, \bar{y})$ ,

$$\bar{x} = x - y_1 \left( \frac{1 + y_1^2}{y_2} \right)$$

$$\bar{y} = y + \left( \frac{1 + y_1^2}{y_2} \right)$$

From (1),



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$$\sqrt{x} + \sqrt{y} = \sqrt{a} \Rightarrow \text{Differentiate w.r.to. } x$$

$$\frac{d}{dx}(\sqrt{x} + \sqrt{y}) = \frac{d}{dx}(\sqrt{a})$$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-1/2}}{y^{-1/2}} = -\frac{y^{1/2}}{x^{1/2}}$$

$$\text{at } P\left(\frac{a}{4}, \frac{a}{4}\right), \frac{dy}{dx} = -1$$

$$y_1 = -1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{-y^{1/2}}{x^{1/2}}\right)$$

$$= -\left[\frac{x^{1/2}\left(\frac{1}{2}y^{-1/2}\right)\left(\frac{dy}{dx}\right) - y^{-1/2}\left(\frac{1}{2}x^{-1/2}\right)}{x}\right]$$

$$= -\frac{1}{2}\left[\frac{x^{1/2}y^{-1/2}\frac{dy}{dx} - y^{-1/2}x^{-1/2}}{x}\right]$$

$$\text{At } P\left(\frac{a}{4}, \frac{a}{4}\right), \frac{d^2y}{dx^2} = \frac{-1}{2}\left[\frac{(1)(-1) - (1)}{\frac{a}{4}}\right]$$



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$$= \frac{-1}{2} \left[ \frac{-2}{\frac{a}{4}} \right]$$

$$y_2 = \frac{4}{a}$$

Consider  $\frac{1+y_1^2}{y_2} = \frac{1+(-1)^2}{\frac{4}{a}} = \frac{2}{\frac{4}{a}} = \frac{a}{2}$

$$\bar{x} = \frac{a}{4} - (-1) \left( \frac{a}{2} \right) = \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}$$

$$\bar{y} = \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}$$

We know that

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\therefore \text{At } P\left(\frac{a}{4}, \frac{a}{4}\right), \rho = \frac{[1+(-1)^2]^{3/2}}{\frac{4}{a}} = (2)^{3/2} \times \frac{a}{4}$$

$$= 2\sqrt{2} \times \frac{a}{4}$$

$$= \frac{a\sqrt{2}}{2}$$

$$\therefore \rho = \frac{a\sqrt{2}}{2}$$

From (2), we have



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$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \left(\frac{a}{\sqrt{2}}\right)^2,$$

**3. Find the equation of the circle of curvature at  $(c, c)$  on  $xy = c^2$ .**

**Soln:**

Given curve:  $xy = c^2$  ----> (1)

Point:  $P(c, c)$

Centre of curvature  $C(\bar{x}, \bar{y})$ ,

$$\bar{x} = x - y_1 \left( \frac{1 + y_1^2}{y_2} \right)$$

$$\bar{y} = y + \left( \frac{1 + y_1^2}{y_2} \right) \text{ where } y_1 = \frac{dy}{dx}; y_2 = \frac{d^2y}{dx^2}.$$

From (1),

$$xy = c^2$$

$$\frac{d}{dx}(xy) = \frac{d}{dx}(c^2)$$

$$x \frac{dy}{dx} + y(1) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

At  $P(c, c)$ ,  $\frac{dy}{dx} = \frac{-c}{c} = -1$

$$y_1 = -1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right)$$



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$$= \frac{d}{dx} \left( \frac{-y}{x} \right)$$

$$= - \left[ \frac{x \frac{dy}{dx} - y(1)}{x^2} \right]$$

$$\therefore \text{At } P(c, c), \frac{d^2 y}{dx^2} = - \left[ \frac{c(-1) - (c)}{c^2} \right]$$

$$= \frac{2c}{c^2}$$

$\therefore$

$$y_2 = \frac{2}{c}$$

$$\text{Consider } \frac{1 + y_1^2}{y_2} = \frac{1 + (-1)^2}{\frac{2}{c}} = \frac{2}{\frac{2}{c}} = c$$

$$\bar{x} = c - (-1)(c) = 2c$$

$$\bar{y} = c + c = 2c$$

We know that

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$\therefore \text{at } P(c, c), \rho = \frac{(1 + (-1)^2)^{3/2}}{\frac{2}{c}} = (2)^{3/2} \times \frac{c}{2}$$

$$= 2\sqrt{2} \times \frac{c}{2}$$



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$$= c\sqrt{2}$$

$$\therefore \rho = c\sqrt{2}$$

$\therefore$  (2) becomes

$$(x - 2c)^2 + (y - 2c)^2 = (c\sqrt{2})^2 .$$