Reduce the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ to diagonal form by orthogonal transformation & find A^3.

Gn. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Step 1 :- To find the char. Eqn.

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 7$; $S_2 = 15$; $S_3 = 9$

$\therefore$ The char. eqn. is $\lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$.

Step 2 :- To find E. values & E. vectors.

$$\lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$$

$\lambda = 1, 3, 3$.

$\therefore$ E. values are $\lambda = 1, 3, 3$

To find E. vectors $(A - \lambda I)x = 0$

$$\left(\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (i): when $\lambda = 1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since (1) & (3) rows are same, by taking cofactor for I row we get

$$\frac{x_1}{\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}}$$

$$\frac{x_1}{2} = -\frac{x_2}{0} = \frac{x_3}{-2}$$

$$\therefore X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii): when $\lambda = 3$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since I & III rows are same, & II row is zero th row, Consider any one eqn. from I & II row, we get

$$x_1 + 0x_2 - x_3 = 0.$$

put $x_1 = 0 \Rightarrow 0x_2 - x_3 = 0$

$$0x_2 = x_3$$

$$\frac{x_2}{1} = \frac{x_3}{0}$$

$\therefore \Rightarrow x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Since given matrix A is symmetric.

Let $x_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be an assumed E. vector as

x_3 is orthogonal to x_1 and x_2.

$x_2^T x_3 = 0$ & $x_3^T x_1 = 0$

Now $x_2^T x_3 = 0$

$\Rightarrow [0 \ 1 \ 0] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$\Rightarrow 0a + b + 0c = 0$. — (1)

Now $x_3^T x_1 = 0$

$\Rightarrow [a \ b \ c] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$

$\Rightarrow a + 0b - c = 0$. — (2)

from (2) $a = c$.

$\Rightarrow \frac{a}{1} = \frac{c}{1}$

$\therefore x_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Step 3:- to check pairwise orthogonal E. vector.

$x_1^T x_2 = [1\ 0\ -1]\begin{bmatrix}0\\1\\0\end{bmatrix} = 0$

$x_2^T x_3 = [0\ 1\ 0]\begin{bmatrix}1\\0\\1\end{bmatrix} = 0$

$x_3^T x_1 = [1\ 0\ 1]\begin{bmatrix}1\\0\\-1\end{bmatrix} = 0.$

$\therefore$ E. vectors are pairwise orthogonal.

Step 4:- to find normalized E. vector

$x_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix} \Rightarrow l(x_1) = \sqrt{1+0+1} = \sqrt{2}$ $\therefore$ N.E.V $x_1 = \begin{bmatrix}1/\sqrt{2}\\0/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$

$x_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow l(x_2) = \sqrt{0+1+0} = \sqrt{1}$ $\therefore$ N.E.V $x_2 = \begin{bmatrix}0/\sqrt{1}\\1/\sqrt{1}\\0/\sqrt{1}\end{bmatrix}$

$x_3 = \begin{bmatrix}1\\0\\1\end{bmatrix} \Rightarrow l(x_3) = \sqrt{1+0+1} = \sqrt{2}$. $\therefore$ N.E.V $x_3 = \begin{bmatrix}1/\sqrt{2}\\0/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

Step 5:-

Normalized Model matrix.

$$N = \begin{bmatrix}1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 0 & 1 & 0\\ -1/\sqrt{2} & 0 & 1/\sqrt{2}\end{bmatrix}$$

Step 6:- to check N is orthogonal

(i.e) $N^T N = N N^T = I$.

$$\text{Now } N^T N = \begin{bmatrix}1/\sqrt{2} & 0 & -1/\sqrt{2}\\ 0 & 1 & 0\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 0 & -1/\sqrt{2}\\ 0 & 1 & 0\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore$ N is orthogonal.

<u>Step 7</u> :- To find $D = N^T A N$

Now $D = N^T A N$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$, the diagonal elts are E. values g