SNS COLLEGE OF TECHNOLOGY

(AN AUTONOMOUS INSTITUTION)

COIMBATORE-35

DEPARTMENT OF MATHEMATICS

23MAT101/ MATRICES AND CALCULUS

UNIT I

MATRIX EIGENVALUE PROBLEM

(Two mark)

1. Find the Characteristic equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

Solution:- Let A=$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

The Characteristic equation is $\lambda^2 - s_1\lambda + s_2 = 0$

$s_1 =$ **Sum of the main diagonal elements** $= 1 + 2 = 3$

$s_2 = |A| = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

Hence the required Characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$

2. Find the Characteristic polynomial of $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

Solution:- The Characteristic polynomial is $|A - \lambda I| = \lambda^2 - s_1\lambda + s_2$

$s_1 =$ **Sum of the main diagonal elements** $= 1 + 3 = 4$

$s_2 = |A| = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3 - 8 = -5$

Hence the required Characteristic polynomial is $\lambda^2 - 4\lambda - 5$

3. The product of two Eigen values of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ ***is*** 16**. Find the third Eigen value.**

Solution:-Let the Eigen values of the matrix A ***be*** $\lambda_1, \lambda_2, \lambda_3$

Given $\lambda_1\lambda_2 = 16$

We know that $\lambda_1\lambda_2\lambda_3 = |A|$**, [Since the product of the Eigen values is equal to the determinant of the matrix]**

$$\lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 32$$

$$16\lambda_3 = 32 \quad [\because \lambda_1\lambda_2 = 16]$$

$$\lambda_3 = \frac{32}{16} = 2$$

4. **The Eigen value of the matrix**

$$\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$$ **are 0 and 1. Find the other Eigen value.**

Solution:- WKT Sum of the Eigen values = Sum of the main diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = 11 + (-2) + (-6)$$

$$0 + 1 + \lambda_3 = 3$$

$$\therefore \lambda_3 = 2$$

Therefore the third Eigen value is 2.

5. **Find the Sum and Product of the Eigen values of the matrix** $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Solution:- WKT Sum of the Eigen values = Sum of the main diagonal elements

=2+2+2=6

Product of the Eigen values = $|A| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 8 - 2 = 6$

6. **One of the Eigen values of the matrix** $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{bmatrix}$ **is -9 Find the other two Eigen values**

Solution:- WKT Sum of the Eigen values = Sum of the main diagonal elements

$$\lambda_1+\lambda_2+\lambda_3=7+(-8)+(-8)$$
$$\lambda_1+\lambda_2-9=-9$$
$$\lambda_1+\lambda_2=0\ldots\ldots\ldots\ldots(1)$$

Product of the Eigen values = $|A|$

7. **The matrix** $A=\begin{bmatrix}1 & 2 & 1\\ 2 & 0 & -2\\ 1 & 2 & 3\end{bmatrix}$ **is singular. One of the Eigen Value is 2. Find the other two Eigen Values.**

Solution:-- Sum of the Eigen Values $=\lambda_1+\lambda_2+\lambda_3$

Sum of the main diagonal elements of A = 1+0+3

WKT, Sum of the Eigen Values=Sum of the main diagonal elements

$$\lambda_1+\lambda_2+\lambda_3=1+0+3$$
$$2+\lambda_2+\lambda_3=4$$
$$\lambda_2+\lambda_3=2$$

Product of the Eigen Values $=|A|$

$$\lambda_1\lambda_2\lambda_3=|A|=\begin{vmatrix}1 & 2 & 1\\ 2 & 0 & -2\\ 1 & 2 & 3\end{vmatrix}$$
$$2\lambda_2\lambda_3=-8$$
$$\lambda_2\lambda_3=-4$$

WKT, x^2- **(Sum of the Eigen Value)** $x+$ **Product of the Eigen values**

$$x^2-2x+(-4)=0$$
$$x^2-2x-4=0$$
$$x=\frac{2\pm\sqrt{20}}{2}=1\pm\sqrt{5}$$
$$\therefore\quad \lambda_2=1+\sqrt{5}\quad ,\lambda_3=1-\sqrt{5}$$

8. **Find the Characteristic equation of the matrix** $\begin{pmatrix}1 & 2\\ 0 & 2\end{pmatrix}$ **and get the Eigen Values.**

Solution:-- Given matrix is a triangular matrix. Hence the Eigen Values are 1,2.

The characteristic of the given matrix is,

$$\lambda^2-(Sum\ of\ the\ Eigen\ value-S_1)\lambda+\Pr oduct\ of\ the\ Eigen\ value-S_2=0$$
$$\Rightarrow\lambda^2-(1+2)\lambda+(1)(2)=0$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$$

9. **If α and β are the Eigen Values of A=$\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$ form the matrix whose Eigen Values are α^3 and β^3**

Solution:-- WKT "If $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots\ldots, \lambda_n$ are the Eigen Values of a matrix A, then A^m has Eigen Values ${\lambda^m}_1, \lambda_2^m, \lambda_3^m, \ldots\ldots\ldots\ldots, \lambda_n^m$ (m being positive integer)"

Let $A = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$ α, β **be the Eigen Values**

Now,

$$A^2 = AA = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 38 & -50 \\ -50 & 138 \end{bmatrix}$$

Hence $A^3 = \begin{bmatrix} 38 & -50 \\ -50 & 138 \end{bmatrix}$ **is the matrix whose Eigen Values be α^3 & β^3**

10. **If 1,1,5 are the Eigen Values of the matrix** $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$**, find the Eigen Values of 5A.**

Solution:-- WKT " If $\lambda_1, \lambda_2, \lambda_3$ be the Eigen Values of A, then $k\lambda_1, k\lambda_2, k\lambda_3$ be the Eigen Values of kA".

$\therefore$ The Eigen Values of 5A are 5,5,25.

11. **If 2,3 are the Eigen Values of** $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$**, find the Eigen Values of a.**

Solution:-- Let $\lambda_1, \lambda_2, \lambda_3$ be the Eigen Values of A.

WKT, Sum of the Eigen Values=Sum of the main diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = 2 + 2 + 2$$

$$2 + 3 + \lambda_3 = 6$$

$$\lambda_3 = 1$$

Product of the Eigen Values $= |A|$

$$\lambda_1\lambda_2\lambda_3 = |A| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix}$$

$$2□3□1 = 8 - 2a$$

$$6 - 8 = -2a$$

$$\therefore a = 1$$

12. **Two Eigen values of** $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -2 \end{bmatrix}$ **are equal and they are double the third.Find the Eigen values of** A^2

Solution:-- Let the third Eigen Value be λ

The remaining Eigen Values are 2 λ,2 λ

WKT, Sum of the Eigen Values = Sum of the main diagonal elements

$$2\lambda + 2\lambda + \lambda = 4 + 3 + (-2)$$

$$5\lambda = 5$$

$$\lambda = 1$$

$\therefore$ **Eigen Values of A are 2,2,1.**

Hence Eigen Values of A^2 **are** $2^2, 2^2, 1^2$ **(i.e) 4,4,1**

13. **Prove that Eigen Values of** $-3A^{-1}$ **are the Values of same as those** $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Solution:-- The Characteristic equation of A is

$$\lambda^2 - s_1\lambda + s_2 = 0$$

s_1 = **Sum of the main diagonal elements** $= 1 + 1 = 2$

$$s_2 = |A| = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3$$

The Characteristic equation is

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 3, -1$$

Hence the Eigen values of A are -1,3.

Hence the Eigen values of A^{-1} **are -1,1/3**

Then the Eigen Values of $-3A^{-1}$ ***are*** $-3(-1), -3\left(\frac{1}{3}\right)$

$ie.,, 3, -1$

Hence Eigen Values of A and -3 A^{-1} are same.

14. **Sum of squares of the Eigen Values of** $A = \begin{bmatrix} 1 & 7 & 5 \\ 0 & 2 & 9 \\ 0 & 0 & 5 \end{bmatrix}$ **is?**

Solution:--The Characteristic equation of the given matrix is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 7 & 5 \\ 0 & 2-\lambda & 9 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)(5-\lambda) = 0$$

$$\therefore \lambda = 1, 2, 5$$

Sum of squares of Eigen Values = $1^2+2^2+5^2=30$

15. State Cayley – Hamilton theorem.

Solution : Every square matrix satisfies its own characteristic equation.

16. Find the Eigen value of Adj A ,if $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution :

Since A is an Triangular matrix,The eigen values of A are 3,4,1.

|A| = 3x4x1=12

$$\frac{\text{adj}\,A}{|A|} = A^{-1} \Rightarrow \text{adj}\,A = A^{-1}\,|A|$$

$=12A^{-1}$

The Eigen value of A^{-1} are $\frac{1}{3}, \frac{1}{4}, 1$

The Eigen value of adj A are $12 \times \frac{1}{3}, 12 \times \frac{1}{4}, 12 \times 1$

$\therefore$ 4,3,12