SNS COLLEGE OF TECHNOLOGY
COIMBATORE-35

DEPARTMENT OF MATHEMATICS

APPLICATIONS

Eigenvalue problems arising from population models (Leslie model)

WHAT IS LESLIE MATRIX MODEL?

- The Leslie matrix (also called the Leslie model) is one of the most well-known ways to describe the growth of populations (and their projected age distribution), in which a population is closed to migration, growing in an unlimited environment.

$$\mathbf{X_{n+1}} = \mathbf{L X_n}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{i-1} \\ x_i \end{bmatrix}_{n+1} = \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_{i-1} & f_i \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{i-1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{i-1} \\ x_i \end{bmatrix}_n$$

PROPERTIES OF LESLIE MODEL

- Age composition initially has an effect on population growth rate, but this disappears over time.
- Over time, population generally approaches a stable age distribution.
- Population projection generally shows exponential growth.

PRECISE CHALLENGE

- Let the oldest age attained by the females in some population be 6 years.Divide the population into 3 age classes of 2 years each. Let the Leslie matrix be

$$L=[l_{jk}] = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}$$

a) What is the number of females in each class after 2,4,6 years if each class initially consists of 500 females?

b) For what initial distribution will the number of females in each class change by same proportion? What is the rate of change?

Solution:

a) Initially $X_{(0)} = \begin{bmatrix} 500 \\ 500 \\ 500 \end{bmatrix}$, $X_0^T = [500 \quad 500 \quad 500]$

After 2 years

$$X_{(2)} = L \times X_{(0)}$$

$$= \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 500 \\ 500 \\ 500 \end{bmatrix}$$

$$X_{(2)} = \begin{bmatrix} 1350 \\ 300 \\ 150 \end{bmatrix}$$

After 4 years,

$$X_{(4)} = L \times X_{(2)}$$

$$= \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 1350 \\ 300 \\ 150 \end{bmatrix}$$

$$= \begin{bmatrix} 750 \\ 810 \\ 90 \end{bmatrix}$$

After 6 years,

$$X_{(6)} = L \times X_{(4)}$$

$$= \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 750 \\ 810 \\ 90 \end{bmatrix}$$

$$= \begin{bmatrix} 1899 \\ 450 \\ 245 \end{bmatrix}$$

b) Distribution Vectors:

$(L-\lambda I)X = 0$, where λ is the rate of change

$$L = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}$$

Characteristic equation $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

Where $D_1 = 0$

$D_2 = -1.38$

$D_3 = 0.072$

$\therefore \lambda^3 - 1.38\lambda - 0.072 = 0$

Eigen Values are $\lambda = 1.2, -1.14, -0.05$

To find Eigen Vectors:

$(L-\lambda I)X = 0$

$$\begin{bmatrix} 0-\lambda & 2.3 & 0.4 \\ 0.6 & 0-\lambda & 0 \\ 0 & 0.3 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

When $\lambda = 1.2$,

$-1.2x_1+2.3x_2+0.4x_3 = 0$ --------- (1)

$0.6x_1-1.2x_2 = 0$ ---------- (2)

$0.3x_2-1.2x_3 = 0$ ---------- (3)

Considering (2) &(3)

$$X = \begin{bmatrix} 1 \\ 0.5 \\ 0.125 \end{bmatrix}$$

Consider, $x+0.5x+0.125x = 1500$

$$1.625x = 1500$$

$$x = 923$$

In class 1, $x = 923$

In class 2, $0.5(x) = 0.5 \times 923 = 462$

In class 3, $0.125(x) = 0.125 \times 923 = 115$

$$\therefore \text{Growth rate} = 1.2$$