SNS COLLEGE OF TECHNOLOGY
COIMBATORE-35

DEPARTMENT OF MATHEMATICS

APPLICATIONS

Eigenvalue problems arising from population models (Leslie model)

WHAT IS LESLIE MATRIX MODEL?

- The Leslie matrix (also called the Leslie model) is one of the most well-known ways to describe the growth of populations (and their projected age distribution), in which a population is closed to migration, growing in an unlimited environment.

$$\mathbf{X_{n+1}} = \mathbf{L}\mathbf{X_n}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{i-1} \\ x_i \end{bmatrix}_{n+1} = \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_{i-1} & f_i \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{i-1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{i-1} \\ x_i \end{bmatrix}_{n}$$

PROPERTIES OF LESLIE MODEL

- Age composition initially has an effect on population growth rate, but this disappears over time.
- Over time, population generally approaches a stable age distribution.
- Population projection generally shows exponential growth.

PRECISE CHALLENGE

- Let the oldest age attained by the females in some population be 6 years.Divide the population into 3 age classes of 2 years each. Let the Leslie matrix be

$$L=[l_{jk}] = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}$$

a) What is the number of females in each class after 2,4,6 years if each class initially consists of 500 females?

b) For what initial distribution will the number of females in each class change by same proportion? What is the rate of change?

Solution:

a) Initially $X_{(0)} = \begin{bmatrix} 500 \\ 500 \\ 500 \end{bmatrix}, X_0^T = [500 \quad 500 \quad 500]$

After 2 years

$$X_{(2)} = L \times X_{(0)}$$

$$= \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 500 \\ 500 \\ 500 \end{bmatrix}$$

$$X_{(2)} = \begin{bmatrix} 1350 \\ 300 \\ 150 \end{bmatrix}$$

After 4 years,

$$X_{(4)} = L \times X_{(2)}$$

$$= \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 1350 \\ 300 \\ 1500 \end{bmatrix}$$

$$= \begin{bmatrix} 750 \\ 810 \\ 90 \end{bmatrix}$$

After 6 years,

$$X_{(6)} = L \times X_{(4)}$$

$$= \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 750 \\ 810 \\ 90 \end{bmatrix}$$

$$= \begin{bmatrix} 1899 \\ 450 \\ 245 \end{bmatrix}$$

b) Distribution Vectors:

$(L-\lambda I)X = 0$, where λ is the rate of change

$$L = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}$$

Characteristic equation $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

Where $D_1 = 0$

$D_2 = -1.38$

$D_3 = 0.072$

$\therefore \lambda^3 - 1.38\lambda - 0.072 = 0$

Eigen Values are $\lambda = 1.2, -1.14, -0.05$

To find Eigen Vectors:

$(L-\lambda I)X = 0$

$$\begin{bmatrix} 0-\lambda & 2.3 & 0.4 \\ 0.6 & 0-\lambda & 0 \\ 0 & 0.3 & 0-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

When $\lambda = 1.2$,

$-1.2x_1+2.3x_2+0.4x_3 = 0$ --------- (1)

$0.6x_1-1.2x_2 = 0$ ---------- (2)

$0.3x_2-1.2x_3 = 0$ ---------- (3)

Considering (2) &(3)

$$X = \begin{bmatrix} 1 \\ 0.5 \\ 0.125 \end{bmatrix}$$

Consider, $x+0.5x+0.125x = 1500$

$$1.625x = 1500$$

$$x = 923$$

In class 1, $x = 923$

In class 2, $0.5(x) = 0.5 \times 923 = 462$

In class 3, $0.125(x) = 0.125 \times 923 = 115$

$\therefore$ Growth rate = 1.2

APPLICATIONS:

STRETCHING OF AN ELASTIC MEMBRANE

ELASTIC DEFORMATION

Elastic deformation is when objects that can change due to stretching, twisting, compression and bending, but once released, they return to their original shape. But, inelastic deformation is when an object can be stretched but cannot return back to its original shape.

ELASTIC DEFORMATION

An elastic membrane in the x_1x_2 plane with boundary circle ${x_1}^2 + {x_2}^2 = 1$ is stretched so that a point P: (x_1, x_2) goes over into the point Q: (y_1, y_2) given by

Y= $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$=AX=$\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in components

$y_1 = 5x_1 + 3x_2$

$y_2 = 3x_1 + 5x_2$

Find the principal directions, that is, the directions of the position vector x of P for which the direction of the position vector y of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

ELASTIC DEFORMATION

We know that Y=AX

NOW Y= λX

Comparing AX= λX

$$(A-\lambda I)X=0$$

The characteristics equation is given by $\lambda^2 - S_1\lambda + S_2 = 0$

Here, S_1=10, $S_2 = 16$

$\therefore$ *The characteristic equation is* $\lambda^2 - 10\lambda + 16 = 0$

$$\therefore \lambda = 2,8$$

To find Eigen Vectors:

$(A-\lambda I)X = 0$

$$\begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

ELASTIC DEFORMATION

When $\lambda = 2$,

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$3x_1 + 3x_2 = 0$ --------- (1)

$3x_1 + 3x_2 = 0$ ---------- (2)

Since both are same equations,

The equation is reduced to a single equation

$$3x_1 + 3x_2 = 0$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

ELASTIC DEFORMATION

When $\lambda = 8$,

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-3x_1 + 3x_2 = 0 \quad \text{---------} \ (1)$$

$$3x_1 - 3x_2 = 0 \quad \text{----------} \ (2)$$

Since both are same equations,

The equation is reduced to a single equation

$$-3x_1 + 3x_2 = 0$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

ELASTIC DEFORMATION

To find the principal direction:

If $\lambda = 8$,

Then i) $\cos^{-1}\left[\frac{1}{\sqrt{1^2+1^2}}\right] = \cos^{-1}\left[\frac{1}{\sqrt{2}}\right] = 45^0$

ii) $\cos^{-1}\left[\frac{1}{\sqrt{1^2+1^2}}\right] = \cos^{-1}\left[\frac{1}{\sqrt{2}}\right] = 45^0$

If $\lambda = 2$,

i) $\cos^{-1}\left[\frac{1}{\sqrt{1^2+(-1)^2}}\right] = \cos^{-1}\left[\frac{1}{\sqrt{2}}\right] = 45^0$

ii) $\cos^{-1}\left[\frac{-1}{\sqrt{1^2+(-1)^2}}\right] = \cos^{-1}\left[\frac{-1}{\sqrt{2}}\right] = 135^0$

ELASTIC DEFORMATION

We thus obtain as eigenvectors of A, for instance, $[1 \quad -1]^T$ corresponding to λ_1 and $[1 \quad 1]^T$ corresponding to λ_2 (or a nonzero scalar multiple of the se). These vectors make 45^0 and 135^0 angles with the positive x_1-direction. They give the principal directions, the answer to our problem. The eigenvalues show that in the principal directions the membrane is stretched by factors 8 and 2, respectively.

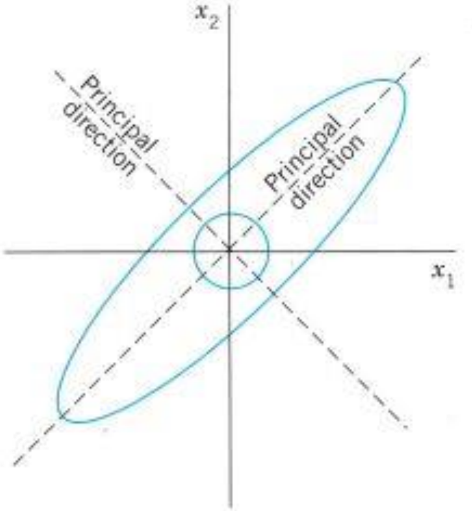
if we set, $x = r\,cos\theta, y = r\,sin\theta$ then a boundary point of the unstretched circular membrane has coordinates $cos\theta, sin\theta$. Hence, after the stretch we have

$$x = 8\,cos\theta, y = 2\,sin\theta$$

ELASTIC DEFORMATION

Since $sin^2\theta + cos^2\theta$=1,

$$\frac{x^2}{8^2} + \frac{y^2}{2^2} = 1$$



This shows that the deformed boundary is a ellipse.


Thank You!