CAYLEY HAMILTON THEOREM

STATEMENT

Every square matrix satisfies its own characteristics equation.

Problems:

1.Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

Solution:

The Characteristic equation is given by

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$$

Where

c_1= Sum of leading diagonal elements

$$= 1 + 2 + 2$$

$$= 5$$

c_2=Sum of the minors of leading diagonal elements.

$$= \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}$$

$$= 4 + 2 + 2$$

$$= 8$$

c_3=det[A]

$$= \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 1(4-0) + 0(4-0) - 2(0)$$

$$= 4$$

Sub the values of c_1, c_2, c_3 in characteristic equation

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$$

We get $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

By Cayley Hamilton theorem matrix A satisfies the equation $A^3 - 5A^2 + 8A - 4I = 0$

$$A^2 = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -14 \\ 14 & 8 & 28 \\ 0 & 0 & 8 \end{bmatrix}$$

Now

$$A^3 - 5A^2 + 8A - 4I = \begin{bmatrix} 1 & 0 & -14 \\ 14 & 8 & 28 \\ 0 & 0 & 8 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -14 \\ 14 & 8 & 28 \\ 0 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 5 & 0 & -30 \\ 30 & 20 & 60 \\ 0 & 0 & 20 \end{bmatrix} + \begin{bmatrix} 8 & 0 & -16 \\ 16 & 16 & 32 \\ 0 & 0 & 16 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley theorem is verified.

1. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence find A^{-1}

Solution:

The Characteristic equation is given by

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$$

Where

c_1= Sum of leading diagonal elements

$$= 2 + 2 + 2$$

$$= 6$$

c_2=Sum of the minors of leading diagonal elements.

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4 - 1) + (4 - 1) + (4 - 1)$$

$$= 9$$

c_3=det[A]

$$= \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= 2(4 - 1) + 1(-2 + 1) + 1(1 - 2)$$

$$= 4$$

Sub the values in characteristic equation

$$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$$

We get $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \rightarrow$ (1)

By Cayley Hamilton theorem matrix A satisfies the equation $A^3 - 6A^2 + 9A - 4I = 0$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & -22 \end{bmatrix}$$

Now

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & -22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & -22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & 30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley theorem is verified.

Computing A^{-1}: Pre multiplying equ (1) by A^{-1}

We get $\quad A^2 - 6A + 9I - 4A^{-1} = 0$

$$4A^{-1} = A^2 - 6A + 9I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$