UNIT III - PARTIAL DIFFERENTIAL EQUATIONS

Formation of partial differential equations – Lagrange's linear equations – Solution of standard types of first order partial differential equations – linear partial differential equations of second order with constant co-efficients (Homogeneous problems)

HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS :

A homogeneous linear partial differential equation is of the form,

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = f(x,y) \quad \text{---}(1)$$

where $a_0, a_1, a_2, \ldots a_n$ are constants.

Here $\frac{\partial}{\partial x} = D$, $\frac{\partial}{\partial y} = D'$

(1) becomes,

$$a_0 D^n z + a_1 D^{n-1} D' z + a_2 D^{n-2} D'^2 z + \cdots + a_n D'^n z = f(x,y) \quad \text{---}(2)$$

Solution of Homogeneous linear PDE :

The complete solution of (2) is

$z =$ Complementary function + particular Integral.

$z = CF + PI$

To find CF :

The CF is the solution of equation,

$$[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n] z = 0$$

In the above equation, put $D \to m$ & $D' \to 1$ then we get,

$$[a_0 m + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n] = 0$$ which is called auxillary equations

For $n=2$, $a_0 m^2 + a_1 m + a_2 = 0$

m_1, m_2 are the two roots for the above equation.

Case (i) : If the roots are real & different (or imaginary), m_1, m_2 (say)
then C.F is $z = f_1(y+m_1 x) + f_2(y+m_2 x)$

Case (ii) : If the roots are real & equal, $m_1 = m_2 = m$ (say)
then C.F is $z = f_1(y+mx) + x f_2(y+mx)$

Hint : RHS is zero then P.I $= 0$

1) Solve: $(D^2 - 5DD' + 6D'^2)z = 0$

Soln:

The Auxiliary equation (AE) is

$m^2 - 5m + 6 = 0$ [Replace $D \to m$, $D' \to 1$]

$m_1 = 3$, $m_2 = 2$

$\Rightarrow$ The roots are real & different.

C.F: $z = f_1(y + m_1 x) + f_2(y + m_2 x)$

$= f_1(y + 3x) + f_2(y + 2x)$

P.I: P.I $= 0$

$\therefore$ solution is $z = C.F + P.I$

$= f_1(y + 3x) + f_2(y + 2x)$

2) Solve: $(D^2 - 6DD' + 9D'^2)z = 0$

Soln: AE is $m^2 - 6m + 9 = 0$ [Replace $D \to m$, $D' \to 1$]

$m_1 = 3$, $m_2 = 3$.

$\Rightarrow m_1 = m_2 = 3 = m$ (say).

$\Rightarrow$ the roots are real & equal.

C.F: $z = f_1(y + mx) + x f_2(y + mx)$

$= f_1(y + 3x) + x f_2(y + 3x)$

P.I: P.I $= 0$

$\therefore$ Solution is $z = C.F + P.I$

$= f_1(y + 3x) + x f_2(y + 3x)$.

3) Solve: $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$

Soln:

Given: $(2D^2 + 5DD' + 2D'^2)z = 0$

A.E. is $2m^2 + 5m + 2 = 0$ [Replace $D \to m$, $D' \to 1$]

$m_1 = -\frac{1}{2}, -2 = m_2$

$\Rightarrow$ the roots are real & different.

C.F: $z = f_1(y - \frac{1}{2}x) + f_2(y - 2x)$

P.I: P.I $= 0$

$\therefore$ Solution is $z = C.F + P.I$

$= f_1(y - \frac{1}{2}x) + f_2(y - 2x)$

To find particular integral (P.I):

Type I : RHS = $f(x,y) = e^{ax+by}$

$$P.I = \frac{1}{\phi(D,D')} e^{ax+by}$$

Replace $D \to a$, $D' \to b$.

then $P.I = \frac{1}{\phi(a,b)} e^{ax+by}$, provided $\phi(a,b) \neq 0$

If $\phi(a,b) = 0$ then differentiate the Dr. w.r.t D' and multiply by x in Nr.

1) Solve: $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y}$.

Soln: Given: $(D^2 - 5DD' + 6D'^2)z = e^{x+y}$.

A.E is $m^2 - 5m + 6 = 0$ [Replace $D \to m$, $D' \to 1$]

$m_1 = 2, 3 = m_2$

$\Rightarrow$ The roots are real & different.

C.F is $z = f_1(y+2x) + f_2(y+3x)$

P.I: $P.I = \frac{1}{D^2 - 5DD' + 6D'^2} \cdot e^{x+y}$

Replace $D \to 1$, $D' \to 1$

$= \frac{1}{1-5+6} \cdot e^{x+y}$

$= \frac{1}{2} e^{x+y}$

$\therefore$ solution is $z = C.F. + P.I.$

$= f_1(y+2x) + f_2(y+3x) + \frac{1}{2}e^{x+y}$.

2) Solve: $(2D^2 - 2DD' + D'^2)z = 2e^{3y} + e^{x+y}$.

Soln: A.E is $2m^2 - 2m + 1 = 0$

$m = \frac{1}{2} \pm \frac{1}{2}i$

$\therefore$ the roots are imaginary & different.

C.F. is $z = f_1(y + (\frac{1}{2} + \frac{1}{2}i)) + f_2(y + (\frac{1}{2} - \frac{1}{2}i))$

P.I :

$$P.I_1 = \frac{1}{2D^2 - 2DD' + D'^2} \cdot 2e^{3y}$$

Replace $D \to 0$ & $D' \to 3$

$$= \frac{1}{0+9} \cdot 2e^{3y}$$

$$= \frac{2}{9} e^{3y}$$

$$P.I_2 = \frac{1}{2D^2 - 2DD' + D'^2} e^{x+y}$$

Replace $D \to 1$ & $D' \to 1$

$$= \frac{1}{2-2+1} e^{x+y}$$

$$= e^{x+y}.$$

$\therefore$ Solution is $z = C.F. + P.I$

$$= f_1\left(y + \left(\tfrac{1}{2} + \tfrac{1}{2}i\right)\right) + f_2\left(y + \left(\tfrac{1}{2} - \tfrac{1}{2}i\right)\right) + \frac{2}{9}e^{3y} + e^{x+y}.$$

3) Solve: $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

Soln: given: $D^2 - 4DD' + 4D'^2 = e^{2x+y}$

A.E is $m^2 - 4m + 4 = 0$ Replace $D \to m$ & $D' \to 1$

$m_1 = 2, 2 = m_2 = m$ (say)

∴ the roots are real & equal.

C.F is $z = f_1(y+2x) + x f_2(y+2x)$

P.I. $\quad P.I_1 = \frac{1}{D^2 - 4DD' + 4D'^2} \cdot e^{2x+y}$

Replace $D \to 2$; $D' \to 1$

$= \frac{1}{4-8+4} \cdot e^{2x+y}$

$= \frac{1}{0} e^{2x+y}$

D.w.r to 'D' in Dr. & multiply by x to Nr.

$= \frac{1}{2D - 4D'} \cdot x e^{2x+y}$

Replace $D \to 2$; $D' \to 1$

$= \frac{1}{4-4} x e^{2x+y}$

$= \frac{1}{0} x e^{2x+y}$

D.w.r. to 'D' in Dr & multiply by 'x' in Nr.

$$P.I = \frac{1}{2} x^2 e^{2x+y}$$

$\therefore$ Solution is $z = C.F + P.I$

$$= f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}$$

4) Solve: $(D^2 - 3DD' + 2D'^2) z = e^{3x+2y}$

5) Solve: $(D^2 - DD' - 20D'^2) z = e^{5x+y}$

6) Solve: $(D^2 + 2DD' + D'^2) z = e^{x-y}$