Convolution Theorem:

Definition: The convolution of two sequences $\{x(n)\}$ and $\{y(n)\}$ is defined as

(i) $\{x(n) * y(n)\} = \sum_{K=-\infty}^{\infty} f(K)\, g(n-K)$ if the sequences are non-causal. and

(ii) $\{x(n) * y(n)\} = \sum_{k=0}^{n} f(k)\, g(n-k)$ if the sequences are causal.

2. The convolution of two functions $f(t)$ and $g(t)$ is defined as $f(t) * g(t) = \sum_{K=0}^{n} f(KT)\, g(n-K)T$, where T is the sampling period.

State and prove convolution theorem on Z-Transform:

Statement:

If $Z[x(n)] = X(z)$ & $Z[y(n)] = Y(z)$ then

$Z\{x(n) * y(n)\} = X(z) \cdot Y(z)$.

proof:

$$Z\{x(n) * y(n)\} = Z\left\{\sum_{K=-\infty}^{\infty} x(K)\, y(n-K)\right\}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{K=-\infty}^{\infty} x(K)\, y(n-K)\right] z^{-n}$$

$$= \sum_{K=-\infty}^{\infty} x(K) \sum_{n=-\infty}^{\infty} y(n-K)\, z^{-n}$$

By changing the order of summation

$$= \sum_{K=-\infty}^{\infty} x(K) \sum_{m=-\infty}^{\infty} y(m)\, z^{-(m+K)} \qquad \left[\because \text{by putting } m = n-K,\ n = m+K\right]$$

$$= \sum_{K=-\infty}^{\infty} x(K)\, z^{-K} \sum_{m=-\infty}^{\infty} y(m)\, z^{-m}$$

$$= X(z) \cdot Y(z).$$

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Problems:

① Find $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$

Soln:

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-a}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right]$$

$$= a^n * a^n$$

$$= \sum_{k=0}^{n} a^{n-k}\cdot a^k \quad \text{by convolution theorem}$$

$$= \sum_{k=0}^{n} a^n = a^n \sum_{k=0}^{n} (1)^k$$

$$= (n+1)\,a^n$$

② Find $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

Soln:

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-b}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n$$

$$= \sum_{k=0}^{n} a^k\, b^{n-k} \quad \text{by Convolution theorem}$$

$$= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$$

$$= b^n\left[1+\frac{a}{b}+\left(\frac{a}{b}\right)^2+\cdots+\left(\frac{a}{b}\right)^n\right]$$

$$= b^n\,\frac{\left[1-\left(\frac{a}{b}\right)^{n+1}\right]}{1-\frac{a}{b}}$$

$$\left\{\because \text{Formula: } a+ar+ar^2+\cdots+ar^n = \frac{a(1-r^n)}{1-r} \text{ if } r<1 \text{ being a G.P.}\right\}$$

$$1+a+a^2+\cdots+a^{n-1} = \frac{a^n-1}{a-1}$$

$$= \frac{b^{n+1}-a^{n+1}}{b-a}$$