



DEPARTMENT OF MATHEMATICS

Z - TRANSFORMS

Introduction: The Z-transform and advanced Z-transform were introduced by E.I. Jury in 1958. The Z-transforms plays an important role in the Communication engineering and control engineering. In Communication engineering, two basic type of signals are encountered. They are continuous time signal and discrete time signal. For the continuous time signal, laplace transform and fourier transforms play important roles. Z-transform plays an important role in discrete time signal analysis.

Z-transform (two-sided or bilateral):

Let $\{x(n)\}$ be a sequence defined for all integers then its Z-transform is defined as,

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

where z is an arbitrary complex number.

Z-transform (One-sided or unilateral):

Let $\{x(n)\}$ be a sequence defined for $n=0,1,2,\dots$ and $x(n)=0$ for $n<0$, then its Z-transform is defined as,

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

where z is an arbitrary complex number.

Z-transform for discrete values of t :

If $f(t)$ is a function defined for discrete values of t where $t = nT$, $n=0,1,2,3,\dots$ T being the sampling period, then Z-transform of $f(t)$ is defined as

$$Z\{f(t)\} = F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}$$



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Problems:

CSA 9/1/13

5th hr: 2, 4, 8, 10, 22

① Prove that $Z[1] = \frac{z}{z-1}$, $|z| > 1$

Solution:

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z(1) = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \left(\frac{z-1}{z}\right)^{-1}$$

$$\left[\because (1-x)^{-1} = 1+x+x^2+\dots \right. \\ \left. \text{Here } x = \frac{1}{z} \text{ where } \left|\frac{1}{z}\right| < 1 \right]$$

$$\left[\begin{array}{l} |z| > 1 \\ \text{or } |z| > 1 \end{array} \right]$$

$$\boxed{Z(1) = \frac{z}{z-1}}, \quad |z| > 1$$

② Prove that $Z[a^n] = \frac{z}{z-a}$ if $|z| > |a|$

Solution:

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} a^n \cdot \frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$

$$= \left(\frac{z-a}{z}\right)^{-1}$$

$$\left[\because (1-x)^{-1} = 1+x+x^2+\dots \right. \\ \left. \text{Here } x = \frac{a}{z} \right]$$

$$\left|\frac{a}{z}\right| < 1 \Rightarrow |a| < |z|$$

$$\text{i.e., } |z| > |a|$$

$$\boxed{Z\{a^n\} = \frac{z}{z-a}}, \quad |z| > |a|$$



③ Prove that $Z(n) = \frac{z}{(z-1)^2}$, $|z| > 1$

Solution:

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z(n) = \sum_{n=0}^{\infty} n z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{n}{z^n}$$

$$= \sum_{n=0}^{\infty} n \left(\frac{1}{z}\right)^n$$

$$= 0 + \left(\frac{1}{z}\right) + 2\left(\frac{1}{z}\right)^2 + \dots$$

$$= \frac{1}{z} \left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \dots \right]$$

$$= \frac{1}{z} \left[\left(1 - \frac{1}{z}\right)^{-2} \right]$$

$$= \frac{1}{z} \left(\frac{z-1}{z}\right)^{-2} = \frac{1}{z} \frac{z^2}{(z-1)^2}$$

$$\left\{ \begin{aligned} \therefore (1-x)^{-2} &= \\ 1 + 2x + 3x^2 + \dots \end{aligned} \right.$$

Here $x = \frac{1}{z}$

$$\left| \frac{1}{z} \right| < 1 \Rightarrow |z| > 1$$

i.e., $|z| > 1$

$$Z(n) = \frac{z}{(z-1)^2}$$

④ Prove that $Z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right)$ if $|z| > 1$, $n > 0$

Solution:

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z\left\{\frac{1}{n}\right\} = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{z}\right)^n$$

$$= \frac{1}{z} + \frac{1}{2} \left(\frac{1}{z}\right)^2 + \frac{1}{3} \left(\frac{1}{z}\right)^3 + \frac{1}{4} \left(\frac{1}{z}\right)^4 + \dots$$

$$= \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \frac{\left(\frac{1}{z}\right)^4}{4} + \dots$$

$$= -\log\left(1 - \frac{1}{z}\right) \left\{ \begin{aligned} \therefore -\log(1-x) &= x + \frac{x^2}{2} + \\ & x = \frac{1}{z}, \left| \frac{1}{z} \right| < 1 \Rightarrow \frac{x^3}{3} + \dots \\ & |z| > 1 \end{aligned} \right.$$



Problems based on $z(1) = \frac{z}{z-1}$ and $z(a^n) = \frac{z}{z-a}$ if $|z| > |a|$

① Find $z(k)$

Soln: $z[1] = \frac{z}{z-1}$

$$z[k] = z[k \cdot 1] = k z[1] \text{ (by linearity property)}$$

$$= k \left[\frac{z}{z-1} \right]$$

② Find $z[(-1)^n]$

Solution: $z[a^n] = \frac{z}{z-a}$

Here $a = -1$

$$z[(-1)^n] = \frac{z}{z+1}$$

③ Find $z\left[\frac{1}{3^n}\right]$

Solution: $z[a^n] = \frac{z}{z-a}$

Here $a = \frac{1}{3}$, $z\left[\frac{1}{3^n}\right] = \frac{z}{z-\frac{1}{3}} = \frac{3z}{3z-1}$

④ Find $z[e^{an}]$

Solution: $z[a^n] = \frac{z}{z-a}$

$$z[e^{an}] = z[(e^a)^n] = \frac{z}{z-e^a} \text{ (Here } a = e^a \text{)}$$

⑤ Find $z[\cos n\theta]$ and $z[\sin n\theta]$

Solution:

Let $a = e^{i\theta}$

$$a^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

We know that $z[a^n] = \frac{z}{z-a}$, $|z| > |a|$

$$z[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}} = \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$z[\cos n\theta + i \sin n\theta] = \frac{z}{(z - \cos\theta) - i \sin\theta}$$



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$$\begin{aligned}
z [\cos n\theta] + i z [\sin n\theta] &= \left[\frac{z}{(z - \cos\theta) - i \sin\theta} \right] \left[\frac{(z - \cos\theta) + i \sin\theta}{(z - \cos\theta) + i \sin\theta} \right] \\
&= \frac{z(z - \cos\theta) + i z \sin\theta}{(z - \cos\theta)^2 + \sin^2\theta} \\
&= \frac{z(z - \cos\theta) + i z \sin\theta}{z^2 - 2z \cos\theta + \cos^2\theta + \sin^2\theta} \\
z [\cos n\theta] + i z [\sin n\theta] &= \frac{z(z - \cos\theta) + i z \sin\theta}{z^2 - 2z \cos\theta + 1}
\end{aligned}$$

Equating the real and imaginary parts, we get

$$z [\cos n\theta] = \frac{z(z - \cos\theta)}{z^2 - 2z \cos\theta + 1}, \quad |z| > 1$$

$$z [\sin n\theta] = \frac{z \sin\theta}{z^2 - 2z \cos\theta + 1}, \quad |z| > 1$$

(b) Find $z [r^n \cos n\theta]$ and $z [r^n \sin n\theta]$

Solution:

$$\text{Let } a = r e^{i\theta}$$

$$a^n = r^n e^{in\theta}$$

$$= r^n [\cos n\theta + i \sin n\theta]$$

$$a^n = r^n \cos n\theta + i r^n \sin n\theta$$

$$\text{We know } z(a^n) = \frac{z}{z - a}$$

$$z[(r e^{i\theta})^n] = \frac{z}{z - r e^{i\theta}} = \frac{z}{z - r(\cos\theta + i \sin\theta)}$$

$$\begin{aligned}
z[r^n \cos n\theta + i r^n \sin n\theta] &= \frac{z}{z - r \cos\theta - i r \sin\theta} \\
&= \frac{z}{z - r \cos\theta - i r \sin\theta} \cdot \frac{(z - r \cos\theta + i r \sin\theta)}{(z - r \cos\theta + i r \sin\theta)} \\
&= \frac{z(z - r \cos\theta) + i z r \sin\theta}{(z - r \cos\theta)^2 - i^2 r^2 \sin^2\theta}
\end{aligned}$$



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$$z [r^n \cos n\theta + i r^n \sin n\theta] = \frac{z(z - r \cos \theta) + i z r \sin \theta}{z^2 - 2zr \cos \theta + r^2 \cos^2 \theta + r^2 \sin^2 \theta}$$

$$= \frac{z(z - r \cos \theta) + i z r \sin \theta}{z^2 - 2zr \cos \theta + r^2}$$

Equating real and imaginary parts,

$$z [r^n \cos n\theta] = \frac{z(z - r \cos \theta)}{z^2 - 2zr \cos \theta + r^2}, \quad |z| > r$$

$$z [r^n \sin n\theta] = \frac{z r \sin \theta}{z^2 - 2zr \cos \theta + r^2}, \quad |z| > r$$

7) Find $z(t)$.

Soln:

$$z \{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$z \{t\} = \sum_{n=0}^{\infty} nT z^{-n} = T \sum_{n=0}^{\infty} n z^{-n}$$

$$= T z(n) = T \frac{z}{(z-1)^2}$$

$$z \{t\} = \frac{Tz}{(z-1)^2}$$

8) Find $z[e^{-at}]$

Soln:

$$z \{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$z [e^{-at}] = \sum_{n=0}^{\infty} e^{-anT} z^{-n}$$

$$= \sum_{n=0}^{\infty} (e^{-aT})^n z^{-n}$$

$$= z [(e^{-aT})^n]$$

$$z [e^{-at}] = \frac{z}{z - e^{-aT}} \quad [\because z(a^n) = \frac{z}{z-a}]$$

9) Find $z[\sin \omega t]$

Soln:



$$\begin{aligned} \text{We know } Z \{ f(t) \} &= \sum_{n=0}^{\infty} f(nT) z^{-n} \\ Z \{ \sin \omega t \} &= \sum_{n=0}^{\infty} \sin(\omega nT) z^{-n} \\ &= \sum_{n=0}^{\infty} \sin n\theta z^{-n} \quad \text{where } \theta = \omega T \\ &= Z(\sin n\theta) \\ &= \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1} \quad (\text{already proved}) \\ Z \{ \sin \omega T \} &= \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1} \quad (\because \theta = \omega T) \end{aligned}$$

Problems:

① Find $Z \left[\cos \frac{n\pi}{2} \right]$

Solution:

$$\checkmark Z \left[\cos n\theta \right] = \frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1}, \quad |z| > 1$$

$$\therefore Z \left[\cos \frac{n\pi}{2} \right] = \frac{z(z - \cos \pi/2)}{z^2 - 2z \cos \pi/2 + 1} = \frac{z(z - 0)}{z^2 - 0 + 1} = \frac{z^2}{z^2 + 1}$$

$$Z \left[\cos \frac{n\pi}{2} \right] = \frac{z^2}{z^2 + 1}, \quad |z| > 1$$

② Find $Z \left[\frac{1}{n(n+1)} \right]$

Solution:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = A(n+1) + Bn$$

$$\text{Put } n=0 \Rightarrow A=1$$

$$n=-1 \Rightarrow B=-1$$

$$\therefore \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$Z \left[\frac{1}{n(n+1)} \right] = Z \left[\frac{1}{n} \right] - Z \left[\frac{1}{n+1} \right]$$



$$z \left[\frac{1}{n(n+1)} \right] = \log \left(\frac{z}{z-1} \right) - z \log \left(\frac{z}{z-1} \right)$$

$$= (1-z) \log \left(\frac{z}{z-1} \right)$$

③ Find $z [e^{at+b}]$

Soln: $e^{at+b} = e^{at} \cdot e^b = e^b \cdot e^{at}$

We know $z [e^{at}] = \frac{z}{z - e^{aT}}$

$$z [e^{at+b}] = z [e^b \cdot e^{at}] = e^b z [e^{at}]$$

$$= e^b \frac{z}{z - e^{aT}} \quad \text{if } |z| > |e^{aT}|$$

④ Find $z [\cos^2 t]$

Soln: $\cos^2 t = \frac{1 + \cos 2t}{2}$

$$z [\cos^2 t] = z \left[\frac{1 + \cos 2t}{2} \right]$$

$$= \frac{1}{2} z [1 + \cos 2t] = \frac{1}{2} z [1] + \frac{1}{2} z [\cos 2t]$$

$$= \frac{1}{2} \frac{z}{z-1} + \frac{1}{2} \frac{z(z - \cos 2T)}{z^2 - 2z \cos 2T + 1}$$

$$= \frac{1}{2} \left[\frac{z}{z-1} + \frac{z(z - \cos 2T)}{z^2 - 2z \cos 2T + 1} \right]$$

⑤ Find $z [\cos^3 t]$

Soln: $\cos^3 t = \frac{1}{4} [3 \cos t + \cos 3t]$

$$z [\cos^3 t] = \frac{1}{4} z [3 \cos t + \cos 3t]$$

$$= \frac{3}{4} z [\cos t] + \frac{1}{4} z [\cos 3t]$$

$$= \frac{3}{4} \left[\frac{z(z - \cos T)}{z^2 - 2z \cos T + 1} \right] + \frac{1}{4} \left[\frac{z(z - \cos 3T)}{z^2 - 2z \cos 3T + 1} \right]$$



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6) Find $z \left[\sin^2 \frac{n\pi}{2} \right]$

Soln: $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$

$$\sin^2 \frac{n\pi}{2} = \frac{1 - \cos n\pi}{2}$$

$$z \left[\sin^2 \frac{n\pi}{2} \right] = z \left[\frac{1 - \cos n\pi}{2} \right] = z \left[\frac{1 - (-1)^n}{2} \right]$$

$$= \frac{1}{2} z \left[1 - (-1)^n \right]$$

$$= \frac{1}{2} \left[z(1) - z(-1)^n \right]$$

$$= \frac{1}{2} \left[\frac{z}{z-1} - \frac{z}{z-(-1)} \right] = \frac{1}{2} \left[\frac{z}{z-1} - \frac{z}{z+1} \right]$$

$$= \frac{1}{2} \left[\frac{z^2 + z - z^2 + z}{z^2 - 1} \right] = \frac{1}{2} \times \frac{2z}{z^2 - 1} = \frac{z}{z^2 - 1}$$

7) Find $z \left[\frac{1}{(n+1)(n+2)} \right]$ $z \left(\frac{1}{n(n-1)} \right)$

Soln:

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} \quad \frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1}$$

$$1 = A(n+2) + B(n+1)$$

put $n = -2$, $-B = 1 \Rightarrow B = -1$

put $n = -1$, $A = 1$

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$z \left[\frac{1}{(n+1)(n+2)} \right] = z \left[\frac{1}{n+1} \right] - z \left[\frac{1}{n+2} \right]$$

$$= z \log \frac{z}{z-1} - \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}$$

$$= z \log \frac{z}{z-1} - \left[\frac{1}{2} + \frac{1}{3} \left(\frac{1}{z} \right) + \frac{1}{4} \left(\frac{1}{z} \right)^2 + \dots \right]$$

$$= z \log \frac{z}{z-1} - z^2 \left[\frac{1}{2} \left(\frac{1}{z} \right)^2 + \frac{1}{3} \left(\frac{1}{z} \right)^3 + \dots \right]$$

$1 = A(n-1) + Bn$
 $n=0 \Rightarrow A = -1$
 $n=1 \Rightarrow B = 1$
 $z \left(\frac{1}{n(n-1)} \right) = z \left(-\frac{1}{n} \right) + z \left(\frac{1}{n-1} \right)$
 $= -\log \left(\frac{z}{z-1} \right) + \frac{1}{z} \log \left(\frac{z}{z-1} \right)$
 $= \log \left(\frac{z-1}{z} \right) + \frac{1}{z} \log \left(\frac{z}{z-1} \right)$



$$\begin{aligned}z \left[\frac{1}{(n-1)(n+2)} \right] &= z \log \frac{z}{z-1} - z^2 \left[-\log \left(1 - \frac{1}{z} \right) - \frac{1}{z} \right] \\&= z \log \frac{z}{z-1} + z^2 \log \left(\frac{z-1}{z} \right) + z \\&= z \log \left(\frac{z}{z-1} \right) - z^2 \log \left(\frac{z}{z-1} \right) + z \\&= (z - z^2) \log \left(\frac{z}{z-1} \right) + z\end{aligned}$$