1

Z - TRANSFORMS

Introduction: The Z-transform and advanced Z-transform were introduced by E.I. Jury in 1958. The Z-transforms plays an important role in the communication engineering and control engineering. In communication engineering, two basic type of signals are encountered. They are continuous time signal and discrete time signal. For the continuous time signal, laplace transform and fourier transforms play important roles. Z-transform plays an important role in discrete time signal analysis.

Z-transform (two-sided or bilateral):

Let $\{x(n)\}$ be a sequence defined for all integers then its Z-transform is defined as,

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is an arbitrary complex number.

Z-transform (One-sided or unilateral):

Let $\{x(n)\}$ be a sequence defined for $n = 0, 1, 2, \ldots$ and $x(n) = 0$ for $n < 0$, then its Z-transform is defined as,

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Where z is an arbitrary complex number.

Z-transform for discrete values of t:

If $f(t)$ is a function defined for discrete values of t where $t = nT$, $n = 0, 1, 2, 3, \ldots$ T being the sampling period, then Z-transform of $f(t)$ is defined as

$$Z\{f(t)\} = F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n}.$$

CSC 'A' 9/1/13
5th hr: 2, 4, 6, 20, 22

Problems:

(1) Prove that $Z[1] = \frac{z}{z-1}, \ |z| > 1$

Solution:

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z(1) = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \left(\frac{z-1}{z}\right)^{-1}$$

$\left[\because (1-x)^{-1} = 1 + x + x^2 + \cdots \right.$
Here $x = \frac{1}{z}$ where $\left|\frac{1}{z}\right| < 1$
$1 < |z|$
or $\left. |z| > 1 \right]$

$$\boxed{Z(1) = \frac{z}{z-1}}, \ |z| > 1$$

(2) Prove that $Z[a^n] = \frac{z}{z-a}$ if $|z| > |a|$

Solution:

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} a^n \cdot \frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$

$$= \left(\frac{z-a}{z}\right)^{-1}$$

$\{\because (1-x)^{-1} = 1 + x + x^2 + \cdots$
Here $x = \frac{a}{z}$
$\left|\frac{a}{z}\right| < 1 \Rightarrow |a| < |z|$
i.e., $|z| > |a|$

$$\boxed{Z\{a^n\} = \frac{z}{z-a}}, \ |z| > |a|$$

(3) Prove that $Z(n) = \frac{z}{(z-1)^2}, \ |z|>1$

Solution:

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z(n) = \sum_{n=0}^{\infty} n z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{n}{z^n}$$

$$= \sum_{n=0}^{\infty} n \left(\frac{1}{z}\right)^n$$

$$= 0 + \left(\frac{1}{z}\right) + 2\left(\frac{1}{z}\right)^2 + \cdots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right]$$

$\left\{ \because (1-x)^{-2} = 1 + 2x + 3x^2 + \cdots \right.$ Here $x = \frac{1}{z}$, $\left|\frac{1}{z}\right| < 1 \Rightarrow 1 < |z|$ i.e., $\left.|z| > 1\right\}$

$$= \frac{1}{z}\left[\left(1 - \frac{1}{z}\right)^{-2}\right]$$

$$= \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2} = \frac{1}{z}\frac{z^2}{(z-1)^2}$$

$$\boxed{Z(n) = \frac{z}{(z-1)^2}}$$

(4) Prove that $Z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right)$ if $|z|>1, \ n>0$

Solution:

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z\left\{\frac{1}{n}\right\} = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{z}\right)^n$$

$$= \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \frac{1}{4}\left(\frac{1}{z}\right)^4 + \cdots$$

$$= \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \frac{\left(\frac{1}{z}\right)^4}{4} + \cdots$$

$$= -\log\left(1 - \frac{1}{z}\right)$$

$\left\{ \because -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \right.$ $x = \frac{1}{z}$, $\left|\frac{1}{z}\right| < 1$, $\left.|z| > 1\right\}$

Problems based on $Z(1) = \frac{z}{z-1}$ and $Z(a^n) = \frac{z}{z-a}$ if $|z|>|a|$

(1) Find $Z(K)$

Soln: $Z[1] = \frac{z}{z-1}$

$Z[K] = Z[K \cdot 1] = K\, Z[1]$ (by linearity property)

$= K\left[\frac{z}{z-1}\right]$

(2) Find $Z[(-1)^n]$

Solution: $Z[a^n] = \frac{z}{z-a}$

Here $a = -1$

$Z[(-1)^n] = \frac{z}{z+1}$

(3) Find $Z\left[\frac{1}{3^n}\right]$

Solution: $Z[a^n] = \frac{z}{z-a}$

Here $a = \frac{1}{3}$, $Z\left[\frac{1}{3^n}\right] = \frac{z}{z-\frac{1}{3}} = \frac{3z}{3z-1}$

(4) Find $Z[e^{an}]$

Solution: $Z[a^n] = \frac{z}{z-a}$

$Z[e^{an}] = Z[(e^a)^n] = \frac{z}{z-e^a}$ (Here $a = e^a$)

(5) Find $Z[\cos n\theta]$ and $Z[\sin n\theta]$

Solution: Let $a = e^{i\theta}$

$a^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

We know that $Z[a^n] = \frac{z}{z-a}$, $|z| > |a|$

$Z[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}} = \frac{z}{z-(\cos\theta + i\sin\theta)}$

$Z[\cos n\theta + i\sin n\theta] = \frac{z}{(z-\cos\theta) - i\sin\theta}$ *

5

$$Z[\cos n\theta] + i\, Z[\sin n\theta] = \left[\frac{z}{(z-\cos\theta)-i\sin\theta}\right]\left[\frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}\right]$$

$$= \frac{z(z-\cos\theta)+iz\sin\theta}{(z-\cos\theta)^2+\sin^2\theta}$$

$$= \frac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$Z[\cos n\theta] + iZ[\sin n\theta] = \frac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+1}$$

Equating the real and imaginary parts, we get

$$Z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}, \quad |z|>1$$

$$Z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}, \quad |z|>1$$

(6) Find $Z[r^n \cos n\theta]$ and $Z[r^n \sin n\theta]$

Solution:

Let $a = re^{i\theta}$

$a^n = r^n e^{in\theta}$

$= r^n[\cos n\theta + i\sin n\theta]$

$a^n = r^n \cos n\theta + i r^n \sin n\theta$

We know $Z(a^n) = \dfrac{z}{z-a}$

$$Z[(re^{i\theta})^n] = \frac{z}{z-re^{i\theta}} = \frac{z}{z-r(\cos\theta+i\sin\theta)}$$

$$Z[r^n\cos n\theta + i r^n \sin n\theta] = \frac{z}{z-r\cos\theta-ir\sin\theta}$$

$$= \frac{z}{z-r\cos\theta-ir\sin\theta}\cdot\frac{(z-r\cos\theta+ir\sin\theta)}{(z-r\cos\theta+ir\sin\theta)}$$

$$= \frac{z(z-r\cos\theta)+izr\sin\theta}{(z-r\cos\theta)^2-i^2r^2\sin^2\theta}$$

$$z\left[r^n\cos n\theta + i r^n \sin n\theta\right] = \frac{z(z-r\cos\theta)+izr\sin\theta}{z^2-2zr\cos\theta+r^2\cos^2\theta+r^2\sin^2\theta}$$

$$= \frac{z(z-r\cos\theta)+izr\sin\theta}{z^2-2zr\cos\theta+r^2}$$

Equating real and imaginary parts,

$$z\left[r^n\cos n\theta\right] = \frac{z(z-r\cos\theta)}{z^2-2zr\cos\theta+r^2}, \quad |z|>r$$

$$z\left[r^n\sin n\theta\right] = \frac{zr\sin\theta}{z^2-2zr\cos\theta+r^2}, \quad |z|>r$$

(7) Find $z(t)$.

Soln:

$$z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\,z^{-n}$$

$$z\{t\} = \sum_{n=0}^{\infty} nT\,z^{-n} = T\sum_{n=0}^{\infty} n\,z^{-n}$$

$$= T\,z(n) = T\frac{z}{(z-1)^2}$$

$$\boxed{z\{t\} = \frac{Tz}{(z-1)^2}}$$

(8) Find $z[e^{-at}]$

Soln:

$$z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\,z^{-n}$$

$$z[e^{-at}] = \sum_{n=0}^{\infty} e^{-anT}\,z^{-n}$$

$$= \sum_{n=0}^{\infty} (e^{-aT})^n\,z^{-n}$$

$$= z\left[(e^{-aT})^n\right]$$

$$\boxed{z[e^{-at}] = \frac{z}{z-e^{-aT}}} \quad \left[\because z(a^n) = \frac{z}{z-a}\right]$$

(9) Find $z[\sin\omega t]$

Soln:

6

We know $Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$

$$Z\{\sin \omega t\} = \sum_{n=0}^{\infty} \sin(\omega nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} \sin n\theta \, z^{-n} \quad \text{where } \theta = \omega T$$

$$= Z(\sin n\theta)$$

$$= \frac{z \sin\theta}{z^2 - 2z\cos\theta + 1} \quad (\text{already proved})$$

$$Z\{\sin \omega T\} = \frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1} \quad (\because \theta = \omega T).$$

Problems:

(1) Find $Z\left[\cos \frac{n\pi}{2}\right]$

Solution:

$$Z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}, \quad |z| > 1$$

$$\therefore Z\left[\cos \frac{n\pi}{2}\right] = \frac{z(z - \cos \pi/2)}{z^2 - 2z\cos \pi/2 + 1} = \frac{z(z-0)}{z^2 - 0 + 1} = \frac{z^2}{z^2+1}$$

$$Z\left[\cos \frac{n\pi}{2}\right] = \frac{z^2}{z^2+1}, \quad |z| > 1$$

(2) Find $Z\left[\frac{1}{n(n+1)}\right]$

Solution:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = A(n+1) + Bn$$

Put $n = 0 \Rightarrow A = 1$

$n = -1 \Rightarrow B = -1$

$$\therefore \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$Z\left[\frac{1}{n(n+1)}\right] = Z\left[\frac{1}{n}\right] - Z\left[\frac{1}{n+1}\right]$$

$$z\left[\frac{1}{n(n+1)}\right] = \log\left(\frac{z}{z-1}\right) - z\log\left(\frac{z}{z-1}\right)$$

$$= (1-z)\log\left(\frac{z}{z-1}\right)$$

(3) Find $z[e^{at+b}]$

Soln:

$$e^{at+b} = e^{at}.e^{b} = e^{b}.e^{at}$$

We know $z[e^{at}] = \dfrac{z}{z-e^{aT}}$.

$$z[e^{at+b}] = z[e^{b}.e^{at}] = e^{b}\, z[e^{at}]$$

$$= e^{b}\frac{z}{z-e^{aT}} \quad \text{if } |z| > |e^{aT}|$$

(4) Find $z[\cos^2 t]$

Soln:

$$\cos^2 t = \frac{1+\cos 2t}{2}$$

$$z[\cos^2 t] = z\left[\frac{1+\cos 2t}{2}\right]$$

$$= \frac{1}{2} z[1+\cos 2t] = \frac{1}{2} z[1] + \frac{1}{2} z[\cos 2t]$$

$$= \frac{1}{2}\frac{z}{z-1} + \frac{1}{2}\frac{z(z-\cos 2T)}{z^2-2z\cos 2T+1}$$

$$= \frac{1}{2}\left[\frac{z}{z-1} + \frac{z(z-\cos 2T)}{z^2-2z\cos 2T+1}\right]$$

(5) Find $z[\cos^3 t]$

Soln:

$$\cos^3 t = \frac{1}{4}[3\cos t + \cos 3t]$$

$$z[\cos^3 t] = \frac{1}{4} z[3\cos t + \cos 3t]$$

$$= \frac{3}{4} z[\cos t] + \frac{1}{4} z[\cos 3t]$$

$$= \frac{3}{4}\left[\frac{z(z-\cos T)}{z^2-2z\cos T+1}\right] + \frac{1}{4}\left[\frac{z(z-\cos 3T)}{z^2-2z\cos 3T+1}\right]$$

(7)

6) Find $Z\left[\sin^2 \frac{n\pi}{2}\right]$

Soln:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sin^2 \frac{n\pi}{2} = \frac{1-\cos n\pi}{2}$$

$$Z\left[\sin^2\frac{n\pi}{2}\right] = Z\left[\frac{1-\cos n\pi}{2}\right] = Z\left[\frac{1-(-1)^n}{2}\right]$$

$$= \frac{1}{2} Z\left[1-(-1)^n\right]$$

$$= \frac{1}{2}\left[Z(1) - Z(-1)^n\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z-(-1)}\right] = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z+1}\right]$$

$$= \frac{1}{2}\left[\frac{z^2+z-z^2+z}{z^2-1}\right] = \frac{1}{2} \times \frac{2z}{z^2-1} = \frac{z}{z^2-1}$$

7) Find $Z\left[\frac{1}{(n+1)(n+2)}\right]$

$$Z\left(\frac{1}{n(n-1)}\right)$$

Soln:

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

$$\frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1}$$

$$1 = A(n-1) + Bn$$

$$n=0 \Rightarrow A = -1$$

$$n=1 \Rightarrow B = 1$$

$$Z\left(\frac{1}{n(n-1)}\right) = Z\left(\frac{-1}{n}\right) + Z\left(\frac{1}{n-1}\right)$$

$$= -\log\left(\frac{z}{z-1}\right) + \frac{1}{z}\log\left(\frac{z}{z-1}\right)$$

$$= \log\left(\frac{z-1}{z}\right) + \frac{1}{z}\log\left(\frac{z}{z-1}\right)$$

$$1 = A(n+2) + B(n+1)$$

put $n = -2$, $-B = 1 \Rightarrow B = -1$

put $n = -1$, $A = 1$

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$Z\left[\frac{1}{(n+1)(n+2)}\right] = Z\left[\frac{1}{n+1}\right] - Z\left[\frac{1}{n+2}\right]$$

$$= z\log\frac{z}{z-1} - \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}$$

$$= z\log\frac{z}{z-1} - \left[\frac{1}{2} + \frac{1}{3}\left(\frac{1}{z}\right) + \frac{1}{4}\left(\frac{1}{z}\right)^2 + \cdots\right]$$

$$= z\log\frac{z}{z-1} - z^2\left[\frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$Z\left[\frac{1}{(n+1)(n+2)}\right] = z\log\frac{z}{z-1} - z^2\left[-\log\left(1-\frac{1}{z}\right)-\frac{1}{z}\right]$$

$$= z\log\frac{z}{z-1} + z^2\log\left(\frac{z-1}{z}\right) + z$$

$$= z\log\left(\frac{z}{z-1}\right) - z^2\log\left(\frac{z}{z-1}\right) + z$$

$$= (z-z^2)\log\left(\frac{z}{z-1}\right) + z$$