Differential equations:

Differential equations are used to represent continous time LTI system. The differential equations relates the i/p & o/p of the systems.

Solving differential equations using Fourier Method.

$$y(t) = x(t) * h(t)$$

$$y(f) = x(f) \cdot h(f)$$

$$\boxed{\therefore H(f) = \frac{y(f)}{x(f)}} \text{ or } H(\omega) = \frac{y(\omega)}{x(\omega)}$$

$H(f)$ (or) $H(\omega) \rightarrow$ system transfer function (or) Frequency response

1. The system produces the o/p $y(t) = e^{-t} \cdot u(t)$ for an input $x(t) = e^{-2t} u(t)$. Determine the impulse response and frequency response of the system.

i/p $x(t) = e^{-2t} u(t)$

o/p $y(t) = e^{-t} \cdot u(t)$

$$x(f) = \frac{1}{2 + j2\pi f} \qquad y(f) = \frac{1}{1 + j2\pi f}$$

$$\therefore H(f) = \frac{y(f)}{x(f)} = \frac{2 + j2\pi f}{1 + j2\pi f}.$$

$$h(t) = F^{-1}[H(f)]$$

$$= F^{-1}\left[\frac{2 + j2\pi f}{1 + j2\pi f}\right]$$

$$= F^{-1}\left[\frac{1 + 1 + j2\pi f}{1 + j2\pi f}\right]$$

$$= F^{-1}[1] + F^{-1}\left[\frac{1}{1 + j2\pi f}\right]$$

$$\boxed{h(t) = \delta(t) + e^{-t} u(t)}$$

② The Differential equation of the system is given as $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = -\frac{d}{dt}x(t)$

Determine freq. response and Impulse response by taking Fourier transform

$$(j\omega)^2 y(\omega) + 5j\omega\, y(\omega) + 6\, y(\omega) = (-j\omega)x(\omega)$$

$$y(\omega)\left[(j\omega)^2 + 5j\omega + 6\right] = -j\omega\, x(\omega)$$

$$H(\omega) = \frac{y(\omega)}{x(\omega)} = \frac{-j\omega}{(j\omega)^2 + 5j\omega + 6}$$

$$H(\omega) = \frac{-j\omega}{(j\omega+2)(j\omega+3)}$$

$$= \frac{2}{j\omega+2} - \frac{3}{j\omega+3}$$

$$\boxed{H(\omega) = 2\cdot\frac{1}{j\omega+2} - 3\cdot\frac{1}{j\omega+3}}$$

$$-j\omega = A(j\omega+3) + B(j\omega+2)$$

put $j\omega = -2$

$$\boxed{2 = A}$$

put $j\omega = -3$

$$+3 = -B$$

$$\boxed{B = -3}$$

Solving differential equation using Laplace transform shifting property of unilateral Laplace transform

$$L\left[\frac{d}{dt}x(t)\right] = s.X(s) - x(0^-)$$

$$L\left[\frac{d^2}{dt^2}x(t)\right] = s^2.X(s) - s\,x(0^-) - x'(0^-)$$

$$L\left[\frac{d^3}{dt^3}x(t)\right] = s^3 X(s) - s^2 x(0^-) - s\,x'(0^-) - x''(0^-)$$

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differential equation

1. solve using differential equation with initial condition $\frac{d}{dt}y(t) + 5y(t) = x(t)$ with initial condition and i/p $x(t) = 3e^{-2t}u(t)$.

$y(0^-) = -2$

$$\frac{d}{dt}y(t) + 5y(t) = x(t)$$

$$s.Y(s) - y(0^-) + 5Y(s) = X(s)$$

$$s.Y(s) + 2 + 5Y(s) = \frac{3}{s+2}$$

$$Y(s)[s+5] + 2 = \frac{3}{s+2}$$

$$Y(s)[s+5] = \frac{3}{s+2} - 2$$

$$Y(s) = \frac{3}{(s+2)(s+5)} - \frac{2}{(s+5)}$$

$$\frac{3}{(s+2)(s+5)} = \frac{A}{s+2} + \frac{B}{s+5}$$

$$3 = A(s+5) + B(s+2)$$

put $s = -5$ $\quad 3 = -3B \Rightarrow \boxed{B = -1}$

put $s = -2$ $\quad 3 = 3A \Rightarrow \boxed{A = 1}$

$$y(s) = \left[\frac{1}{s+2} - \frac{1}{s+5}\right] - \frac{2}{s+5}$$

$$y(s) = \frac{1}{s+2} - \frac{3}{s+5}$$

$$\boxed{y(t) = e^{-2t}u(t) - 3e^{-5t}u(t)}$$

System Transfer function

$$y(t) = x(t) * h(t)$$

$$y(s) = X(s) * H(s)$$

$$\boxed{H(s) = \frac{y(s)}{x(s)}} \rightarrow \text{system Transfer function.}$$

Frequency Response

substitute $\boxed{s = j\omega}$ in $H(s)$, we get frequency response $H(j\omega)$

1. The input, output relation of a system at initial state is given by $\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = \frac{d}{dt}x(t) + 2x(t)$. Find system transfer function, Frequency response and Impulse response.

$$s^2 y(s) + 4s \cdot y(s) + 3y(s) = s\,x(s) + 2x(s)$$

$$y(s)\left[s^2 + 4s + 3\right] = x(s)\left[s+2\right]$$

$$H(s) = \frac{y(s)}{x(s)} = \frac{s+2}{s^2+4s+3}$$

Freq. response

$$H(j\omega) = \frac{y(j\omega)}{x(j\omega)} = \frac{j\omega+2}{(j\omega)^2 + 4j\omega + 3}$$