Lattices and Boolean Algebra

Relation:

Relation is called a binary operation between a pair of objects.

Properties of Relation:

Reflexive:

Let R be a relation on set X

R is reflexive if xRx, $\forall x \in X$.

Symmetric:

R is symmetric if $xRy \Rightarrow yRx$, $\forall x, y \in X$.

Transitive:

R is Transitive if xRy and $yRz \Rightarrow xRz$, $\forall x, y, z \in X$.

Antisymmetric:

R is antisymmetric if xRy and $yRx \Rightarrow x = y$

Eg:

Let $A = \{1, 2, 3\}$ and R be the relation "$\leq$"

$$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

$\therefore$ R is reflexive, Transitive, Antisymmetric.

Equivalence Relation:

A relation R on X is called an equivalence relation on X if R satisfies reflexive, symmetric and Transitive.

partial order Relation:

A relation R which satisfies reflexive, antisymmetric and transitive is called an partial order relation.

Partially ordered set or Posets

A set together with a partial order relation defined on it is called partially ordered set. It is denoted by $(P, \leq)$

Hasse diagram:

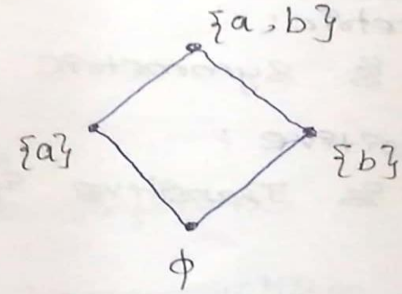
A graphical representation of a poset is called Hasse diagram.

1. Draw Hasse diagram for $(P(A), \subseteq)$ where
i). $A = \{a, b\}$ ii). $A = \{a, b, c\}$

Soln:

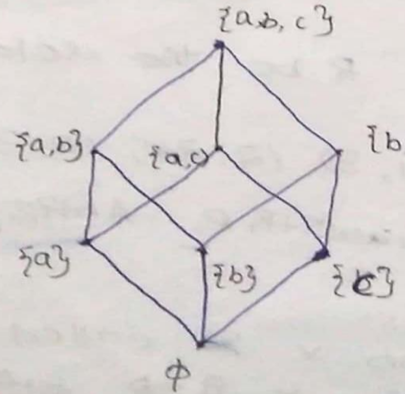
i). $A = \{a, b\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$



ii). $A = \{a, b, c\}$

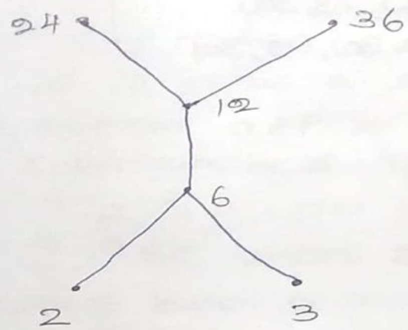
$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$



2. If $x = \{2, 3, 6, 12, 24, 36\}$ and the relation "$\leq$" is defined as $a \leq b$ if a/b. Draw the Hasse diagram of $(x, \leq)$.

Soln.

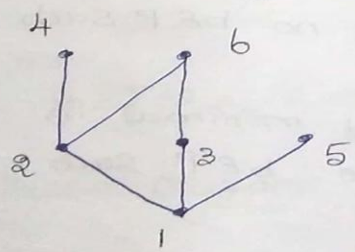
$R = \{(2,6), (2,12), (2,24), (2,36), (3,6), (3,12), (3,24), (3,36), (6,12), (6,24), (6,36), (12,24), (12,36)\}$



3). Draw Hasse diagram where $X = \{1,2,3,4,5,6\}$ and "/" is a partial ordering relation on X.

Soln.

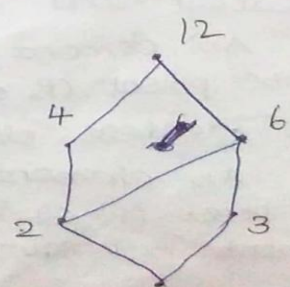
$R = \{(1,2), (1,3), (1,4), (1,5), (1,6), (2,4), (2,6), (3,6)\}$



4). If $X = \{1,2,3,4,6,12\}$ and "$\leq$" is the relation such that $x \leq y$ iff x/y.

Soln.

$R = \begin{cases} (1,2), (1,3), (1,4), (1,6), (1,12) \\ (2,4), (2,6), (2,12), (3,6), (3,12) \\ (4,12), (6,12) \end{cases}$

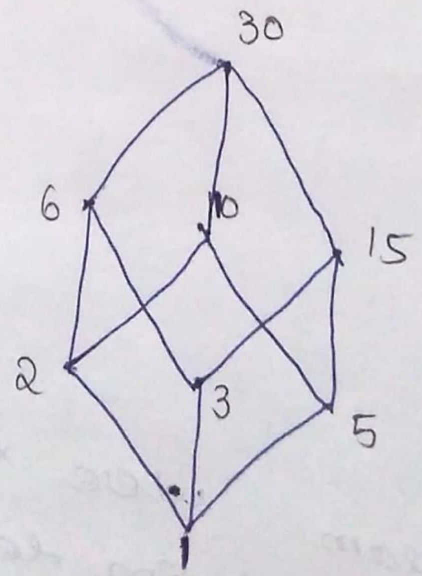


5). Draw the Hasse Diagram for D_{30} (Divisors of 30) $(D_{30}, /)$

Soln.

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$R = \{(1,2), (1,3), (1,5), (1,6), (1,10), (1,15), (1,30), (2,6), (2,10), (2,30), (3,6), (3,15), (3,30), (5,10), (5,15), (5,30), (6,30), (10,30), (15,30)\}$



Maximal and Minimal Element:

An element $a \in P$ is called maximal in the poset $(P, \leq)$ if there is no $b \in P$ such that $b > a$.

An element $a \in P$ is called minimal in the poset $(P, \leq)$ if there is no $b \in P$ such that $b < a$.

Greatest and Least element:

An element 'a' is the greatest element of the poset $(P, \leq)$ if $b \leq a$ for all $b \in P$. The greatest element is unique if it exist.

An element 'a' is the least element of the poset $(P, \leq)$ if $a \leq b$ for all $b \in P$. The least element is unique if it exist.

Upper bound and Lower Bound

Let $(P, \leq)$ be a poset and $A \leq P$.
Any elt $x \in P$ is an upper bound if for all $a \in A$ such that and $a \leq x$.
Similarly, an elt $x \in P$ is a lower bound if for $a \in A$ and $x \leq a$.

Least upper bound:

Let $(P, \leq)$ be a poset and $A \subseteq P$.
An element $a \in P$ is said to be least upper bound or supremum of a if
i). a is a upper bound of A
ii). $a \leq c$, where c is any other upper bound of A.

Greatest lower bound:

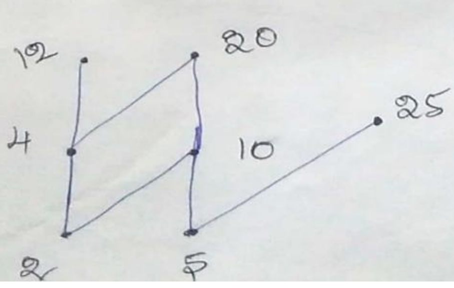
Let $(P, \leq)$ be a poset and $A \leq P$.
An element $b \in P$ is said to be greatest lower bound or infimum of A if
i) b is a lower bound of A
ii) $b \geq d$, where d is any other lower bound of A.

1. Which elements of the poset $\{2, 4, 5, 10, 12, 20, 25\}$ are maximal and which are minimal?

Soln.

$$R = \{(2,4), (2,10), (2,12), (2,20), (4,12), (4,20), (5,10), (5,20), (5,25), (10,20)\}$$

Hasse Diagram:



maximal elements:
12, 20, 25
minimal elements:
2, 5

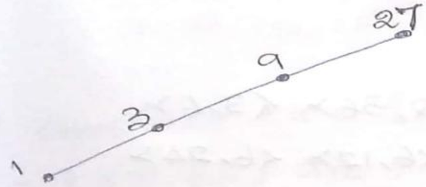
2]. which is the greatest element and a least element in the poset $(P(A), \subseteq)$ where A is any finite set.

Soln.

$\phi \to$ least element

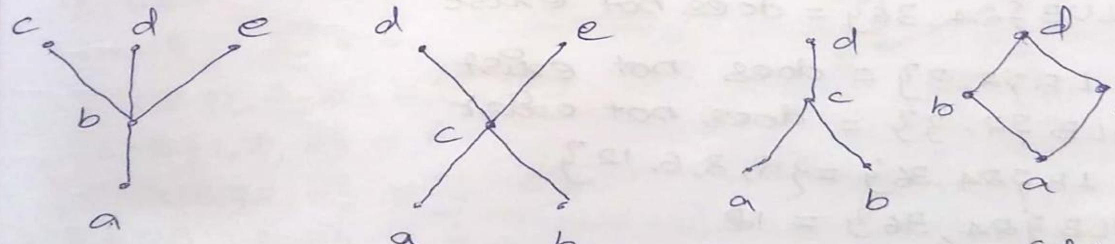
$A \to$ greatest element.

3. Determine whether the poset represented by the hasse diagram of the following have a least, greatest, maximal & minimal.



max elt = 27
min. elt = 1
least elt = 1
Greatest elt = 27

1. Determine whether the posets represents by each of the HD in the following figure have a greatest elt. and a least elt.



(a) (b) (c) (d)

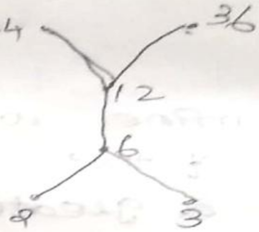
Soln

(a) least elt. is a
greatest elt. does not exist
(b) neither a least nor a greatest elt.
(c) least elt. does not exist
greatest elt. is d
(d) least elt. is a
greatest elt. is d.

5]. Consider $X = \{2, 3, 6, 12, 24, 36\}$ and $R = \{\langle a, b \rangle, a/b\}$. Find LUB and GLB of (2,3) and (24, 36).

Soln.

$R = \{ \langle 2,6 \rangle \langle 2,12 \rangle, \langle 2,24 \rangle \langle 2,36 \rangle \langle 3,6 \rangle \langle 3,12 \rangle \langle 3,24 \rangle \langle 3,36 \rangle \langle 6,12 \rangle \langle 6,24 \rangle \langle 6,36 \rangle \langle 12,24 \rangle \langle 12,36 \rangle \}$



I). $UB\{2,3\} = \{6, 12, 24, 36\}$

$LUB\{2,3\} = 6$

$UB\{24, 36\}$ = does not exist

$LUB\{24, 36\}$ = does not exist

II). $LB\{2,3\}$ = does not exist

$GLB\{2,3\}$ = does not exist

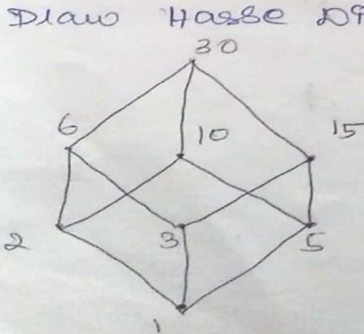
$LB\{24,36\} = \{2, 3, 6, 12\}$

$GLB\{24,36\} = 12$

6]. Let $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ with a relation $x \leq y$ iff x divides y.

Find:
i). All lower bounds of 10 and 15.
ii). GLB of 10 and 15
iii). All upper bound ~~of~~ 10 and 15
iv). LUB of 10 and 15
v). Draw Hasse Diagram for D_{30}.

Soln.



i). $LB(10, 15) = \{1, 5\}$

ii). $GLB(10, 15) = 5$

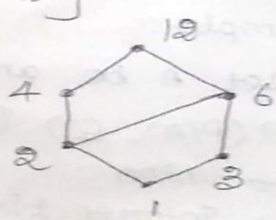
iii). $UB(10, 15) = 30$

$LUB(10, 15) = 30$

7. consider $X = \{1, 2, 3, 4, 6, 12\}$ and $R = \{\langle a, b\rangle \mid a/b\}$. Find LUB and GLB for the Poset (X, R).

Soln.

$R = \{(1,2), (1,3), (1,4), (1,6), (1,12), (2,4), (2,6), (2,12), (3,6), (3,12), (4,12), (6,12)\}$



i). $UB\{1,3\} = \{3, 6, 12\}$

$LUB\{1,3\} = 3$

$UB\{1,2,3\} = \{6, 12\}$

$LUB\{1,2,3\} = 6$

$UB\{2,3\} = \{6, 12\}$

$LUB\{2,3\} = 6$

ii). $LB\{1,3\} = 1$

$GLB\{1,3\} = 1$

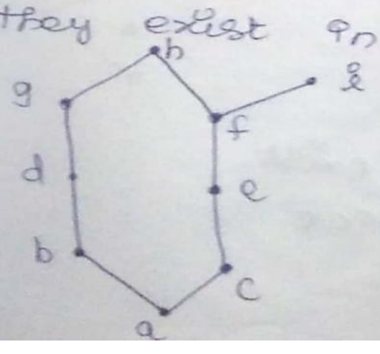
$LB\{1,2,3\} = 1$

$GLB\{1,2,3\} = 1$

$LB\{2,3\} = 1$

$GLB\{2,3\} = 1$

8. Find the GLB & LUB of $\{b, d, g\}$, if they exist in the poset given below.



Soln.

$UB\{b, d, g\} = g, h, i$

$LUB\{b, d, g\} = g$

$LB\{b, d, g\} = a, b$

$GLB\{b, d, g\} = b$