Duality in lattice:

When "$\leq$" is a partial order relation on a set S, then it converse "$\geq$" is also partial order relation on S.

Distributive lattice:

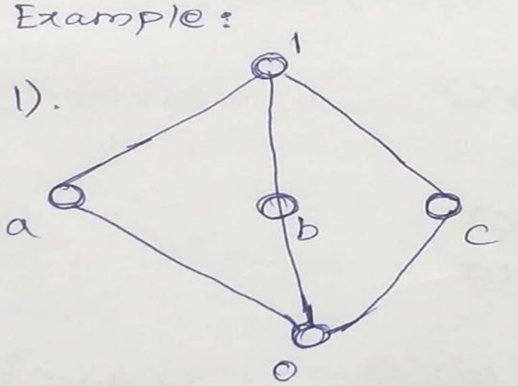
A lattice $(L, \wedge, \vee)$ is said to be distributive lattice if $\wedge$ and $\vee$ satisfies the following conditions, $\forall a, b, c \in L$.

D_1 : $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

D_2 : $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Example:

1).



$L = \{0, a, b, c, 1\}$

Here

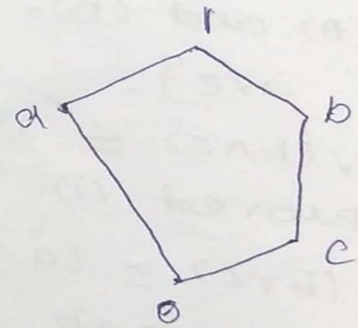
$a \vee (b \wedge c) = a \vee 0 = a$

$(a \vee b) \wedge (a \vee c) = 1 \wedge 1 = 1$

since $a \neq 1$

It is Not a Distributive lattice

2).



$L = \{0, a, b, c, 1\}$

Here

$c \wedge (a \vee b) = c \wedge 1$
$= a$

$(c \wedge a) \vee (c \wedge b) = 0 \vee c$
$= c$

$c \wedge (a \vee b) \neq (c \wedge a) \vee (c \wedge b)$

Not a distributive lattice.

Theorem 1:
Prove that any chain is a distributive lattice.

Proof:
Let $(L, \wedge, \vee)$ be a given chain and $\forall a, b \in L$.
Since any 2 elts. of a chain are comparable, we've either $a \leq b$ or $b \leq a$.

Case 1: $a \leq b$	Case 2: $b \leq a$
Then $GLB\{a, b\} = a$	Then $GLB\{a, b\} = b$
$LUB\{a, b\} = b$	$LUB\{a, b\} = a$

In both cases, any 2 elts. of a chain has both GLB and LUB.
$\therefore$ Any chain is a lattice.
Next we prove $(L, \wedge, \vee)$ satisfies distributive property.
Let $a, b, c \in L$.
Since any chain satisfies comparable property, we've the following 6 cases.

Case 1: $a \leq b \leq c$
2: $a \leq c \leq b$
3: $b \leq a \leq c$
4: $b \leq c \leq a$
5: $c \leq a \leq b$
6: $c \leq b \leq a$

Case 1: $a \leq b \leq c$
Prove D_1: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

LHS
$a \vee (b \wedge c)$
$\Rightarrow a \vee b$ $(\because b \leq c)$
$\Rightarrow b$ $(\because a \leq b)$

RHS
$(a \vee b) \wedge (a \vee c)$
$\Rightarrow b \wedge c$
$\Rightarrow b$

LHS = RHS

$\therefore$ D_1 condition is true for the case 1.
Similarly, we can easily prove D_1 - property to the remaining 5 cases.

$\therefore$ $(L, \wedge, \vee)$ is a distributive lattice.

$\therefore$ Any chain is a distributive lattice.

Theorem : 2 Modular inequality

~~Let~~ If $(L, \wedge, \vee)$ is a lattice, then for any $a, b, c \in L$, $a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$

Proof:

Assume $a \leq c \longrightarrow (1)$

$\therefore \quad a \vee c = c$

By distributive inequality,

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$

$\Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$ (using (1))

$a \leq c \Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c \longrightarrow (2)$

~~Now~~ Conversely, assume

$$a \vee (b \wedge c) \leq (a \vee b) \wedge c$$

Now, by the defn. of LUB and GLB, we've

$$a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$$

$$\Rightarrow a \leq c$$

$\therefore \quad a \vee (b \wedge c) \leq (a \vee b) \wedge c \Rightarrow a \leq c \rightarrow (3)$

From (2) and (3), we've

$$a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$$

Problem

1. In any distributive lattice $(L, \wedge, \vee)$, $\forall a,b,c \in L$. prove that $a \vee b = a \vee c$, $a \wedge b = a \wedge c \Rightarrow b = c$

Soln.:

$b = b \vee (b \wedge a)$ (Absorption law)

$= b \vee (a \wedge b)$

$= b \vee (a \wedge c)$ Gvn. condition

$= (b \vee a) \wedge (b \vee c)$

$= (a \vee b) \wedge (b \vee c)$

$= (a \vee c) \wedge (b \vee c)$ Gvn. condition

$= (a \wedge b) \vee c$

$= (a \wedge c) \vee c$ Gvn. condition

$= \cancel{a \wedge} (c \wedge a) \vee c$

$b = c$ Absorption law