Subgroups:

Defn:

Let $(G, *)$ be a group. Then $(H, *)$ is said to be subgroup of $(G, *)$ if $H \subseteq G$ and $(H, *)$ itself is a group under the operation $*$.

ie, $(H, *)$ is said to be a subgroup of $(G, *)$ if

i). $e \in H$

ii). For any $a \in H$, $a^{-1} \in H$

iii). For $a, b \in H$, $a * b \in H$

Theorem : 1

The necessary and sufficient condition that a non empty subset H of a group G to be a subgroup is $a, b \in H \Rightarrow a * b^{-1} \in H$.

Proof:

Necessary condition:

Assume that H is a subgroup of G.

To prove $a * b^{-1} \in H$

Let $a, b \in H \Rightarrow b^{-1} \in H$ (Inverse)

Then $a * b^{-1} \in H$

Sufficient condition:

Assume that $a, b \in H \Rightarrow a * b^{-1} \in H \rightarrow (1)$

To prove H is a subgroup of G.

i). closure:

Let $a, b \in H$

Since $b \in H \Rightarrow b^{-1} \in H$

Let $a, b^{-1} \in H \Rightarrow a * (b^{-1})^{-1} \in H$ by (1)

$a * b \in H \Rightarrow H$ is closed.

ii) Identity:

Let $a \in H \Rightarrow a^{-1} \in H$

Then $a * a^{-1} \in H$

$\Rightarrow e \in H$ [$G \rightarrow$ group]

Hence the identity elt. $e \in H$.

iii). Inverse:

Let $e, a \in H$

$\Rightarrow e * a^{-1} \in H$ by (1)

$a^{-1} \in H$

Hence H is a subgroup of G since H itself is a group.

Theorem: 2

The intersection of 2 subgroups of a group is also a subgroup of the group.

(or)

Let G be a group and H_1 and H_2 are subgroups of G. Then $H_1 \cap H_2$ is also a subgroup of G.

Proof:

Let H_1 and H_2 be the two subgroups of G.

To prove $H_1 \cap H_2$ is a subgroup of G.

$\therefore H_1 \cap H_2 \neq \phi$ [$\because$ atleast the identity elt. is present in H_1 and H_2]

Let $a, b \in H_1 \cap H_2$

$\Rightarrow a, b \in H_1$ and $a, b \in H_2$

$\Rightarrow a * b^{-1} \in H_1$ and $a * b^{-1} \in H_2$ [since H_1 and H_2 are subgroups]

$\Rightarrow a * b^{-1} \in H_1 \cap H_2$

For $a, b \in H_1 \cap H_2$, we've $a * b^{-1} \in H_1 \cap H_2$.

$\therefore H_1 \cap H_2$ is a subgroup [By above Theorem].

Theorem: 3

The union of two subgroups of a group need not be a subgroup.

Theorem : 4

The union of two subgroups of a group G is a subgroup iff one is contained in the other.

Proof:

Assume H and K are two subgroups of G and $H \subseteq K$ or $K \subseteq H$.

$\therefore H \cup K = K$ or $H \cup K = H$

Hence $H \cup K$ is a subgroup.

Conversely,

Suppose $H \cup K$ is a subgroup of G.

To prove $H \subseteq K$ or $K \subseteq H$.

Suppose that H is not contained in K and K is not contained in H.

Then $\exists$ elts. $a, b \ni a \in H$ and $a \notin K \rightarrow (1)$

$b \in K$ and $b \notin H \rightarrow (2)$

Clearly, $a, b \in H \cup K$ since $H \cup K$ is a subgroup of G.

$ab \in H \cup K$

Hence $a * b \in H$ or $a * b \in H$

case 1). Let $a * b \in H$

Since, $a \in H \Rightarrow a^{-1} \in H$

Hence $a^{-1} * (a * b) \in H$

$(a^{-1} * a) * b \in H$ Associative

$e * b \in H$

$b \in H$ which is a contradiction to our assumption

Case 2). let $a * b \in K$

Since $b \in K \Rightarrow b^{-1} \in K$

Hence $b^{-1} * (a * b) \in K$

$b^{-1} * (b * a) \in K$

$(b^{-1} * b) * a \in K$

$e * a \in K \Rightarrow a \in K$ which is a $\Rightarrow \Leftarrow$ to our assumption.

$\therefore$ our assumption is wrong.