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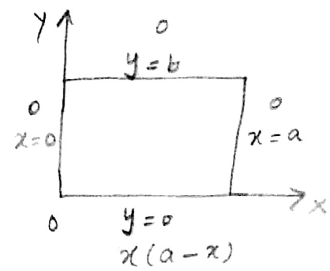
TYPE 3 : Rectangular plate : Temperature given in horizontal edge :

1) Find the steady state temperature distribution in a rectangular plate of sides a and b insulated at the lateral surface and satisfying the boundary conditions $u(0,y) = u(a,y) = 0$ for $0 \le y \le b$, $u(x,b) = 0$ and $u(x,0) = x(a-x)$ for $0 \le x \le a$.



Solution :

Step 1 : The steady state two dimension equation is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Step 2 : The boundary conditions are

(i) $u(0,y) = 0$

(ii) $u(a,y) = 0$

(iii) $u(x,b) = 0$

(iv) $u(x,0) = x(a-x)$

Step 3 : The suitable solution is given by,

$$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \rightarrow (1)$$

Step 4 : Applying condition (i) in (1),

$$u(0,y) = c_1 (c_3 e^{py} + c_4 e^{-py}) = 0$$

Here $c_3 e^{py} + c_4 e^{-py} \neq 0$ [$\because$ It is defined for all y]

$$\therefore c_1 = 0$$

Subs $c_1 = 0$ in (1);

$$u(x,y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \rightarrow (2)$$

Step 5 : Applying condition (ii) in (2),

$$u(a,y) = c_2 \sin pa (c_3 e^{py} + c_4 e^{-py}) = 0$$

Here $c_3 e^{py} + c_4 e^{-py} \neq 0$ [$\because$ It is defined for all y]

$c_2 \neq 0$ [$\because c_1 = 0$ we get a trivial solution]

$\therefore \sin pa = 0 = \sin n\pi$

$$pa = n\pi$$

$$\boxed{p = \frac{n\pi}{a}}$$

subs $p = \frac{n\pi}{a}$ in (2),

$$u(x,y) = c_2 \sin \frac{n\pi x}{a} \left(c_3 e^{n\pi y/a} + c_4 e^{-n\pi y/a} \right) \rightarrow (3)$$

Step 6: Applying Condition (iii) in (3),

$$u(x,b) = c_2 \sin\left(\frac{n\pi x}{a}\right) \left[c_3 e^{n\pi b/a} + c_4 e^{-n\pi b/a} \right]$$

Here $c_2 \neq 0$ [$\because c_1 = 0$ we get a trivial solution]

$\sin\left(\frac{n\pi x}{a}\right) \neq 0$ [$\because$ It is defined for all x]

$$c_3 e^{n\pi b/a} + c_4 e^{-n\pi b/a} = 0$$

$$c_3 e^{n\pi b/a} = -c_4 e^{-n\pi b/a} = \frac{-c_4}{e^{n\pi b/a}}$$

$$\boxed{c_4 = -c_3 e^{2n\pi b/a}}$$

subs the value of c_4 in (3),

$$u(x,y) = c_2 \sin\left(\frac{n\pi x}{a}\right) \left[c_3 e^{n\pi y/a} - c_3 e^{-n\pi y/a} e^{2n\pi b/a} \right]$$

$$= c_2 c_3 \sin\left(\frac{n\pi x}{a}\right) \left[e^{n\pi y/a} - \frac{e^{n\pi b/a}}{e^{-n\pi b/a}} \cdot e^{-n\pi y/a} \right]$$

$$= c_0 \sin\left(\frac{n\pi x}{a}\right) \left[\frac{e^{-\frac{n\pi}{a}(b-y)} - e^{\frac{n\pi}{a}(b-y)}}{e^{-n\pi b/a}} \right]$$

$$= -c_0 e^{n\pi b/a} \sin\left(\frac{n\pi x}{a}\right) \left[e^{+\frac{n\pi}{a}(b-y)} - e^{-\frac{n\pi}{a}(b-y)} \right]$$

$$= -2c_0 e^{n\pi b/a} \sin\left(\frac{n\pi x}{a}\right) \sinh (b-y)\frac{n\pi}{a}$$

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Step 7 : The most general solution is ,

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \sinh (b-y)\frac{n\pi}{a} \longrightarrow (4)$$

where $c_n = -2c_0 e^{n\pi b/a}$

Step 8 : Applying condition (iv) in (4) , we get

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right) = x(a-x)$$

$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) = x(a-x) \longrightarrow (5)$$

where $B_n = c_n \sinh\left(\frac{n\pi b}{a}\right)$

Expand $f(x)$ as a half range Fourier sine series in $(0,a)$.

$$x(a-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \longrightarrow (6)$$

where $b_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$

From (5) & (6) , $\boxed{B_n = b_n}$

$$B_n = \frac{2}{a}\int_0^a x(a-x) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a}\left\{-x(a-x)\frac{a}{n\pi}\cos\frac{n\pi x}{a} + (a-2x)\frac{a^2}{n^2\pi^2}\sin\frac{n\pi x}{a} - \frac{2a^3}{n^3\pi^3}\cos\frac{n\pi x}{a}\right\}_0^a$$

$$= \frac{2}{a}\left\{-a(a-a)\frac{a}{n\pi}\cos n\pi - \frac{2a^3}{n^3\pi^3}\cos n\pi + \frac{2a^3}{n^3\pi^3}\right\}$$

$u = x(a-x)$
$u' = a-2x$
$u'' = -2$
$v = \sin\frac{n\pi x}{a}$
$v_1 = \frac{-\cos\frac{n\pi x}{a}}{n\pi/a}$
$v_2 = \frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/a^2}$
$v_3 = \frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/a^3}$

$$B_n = \frac{2}{a} \cdot \frac{2a^3}{n^3 \pi^3} \left[1 - (-1)^n\right]$$

$$\boxed{B_n = \frac{4a^2}{n^3 \pi^3} \left[1 - (-1)^n\right]}$$

$$B_n = \begin{cases} \dfrac{8a^2}{n^3 \pi^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Step 9: subs the value of B_n in (4),

$$u(x,y) = \sum_{n=1}^{\infty} \frac{4a^2 \left[1 - (-1)^n\right]}{n^3 \pi^3 \sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi (b-y)}{a}\right)$$