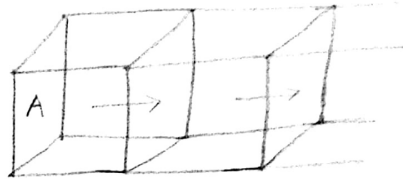




THE ONE-DIMENSIONAL HEAT EQUATION

Derivation of one-dimensional heat equation:

Consider a long thin bar (or wire (or) rod) of constant cross sectional area A and homogeneous conducting material. Let ρ be the density of the material, c be a specific heat and k be the thermal conductivity of the material. We assume that the surface of the bar is insulated so that the heat flow along parallel lines which are perpendicular to the area A .



One dimensional heat equation:

The one-dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Where $\alpha^2 = \frac{k}{\rho c} = \frac{\text{Thermal conductivity}}{\text{Density} \times \text{specific heat}}$

Various Solutions of one-dimensional heat equation:

The various solutions of one-dimensional heat equation

- is
- (i) $u(x, t) = (c_1 e^{px} + c_2 e^{-px}) e^{-\alpha^2 p^2 t}$
 - (ii) $u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-\alpha^2 p^2 t}$
 - (iii) $u(x, t) = (c_1 x + c_2) c_3$

The most suitable solution is

$$u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-\alpha^2 p^2 t}$$



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TYPE I : Problems with zero Boundary Values [i.e., the temperatures at the ends of the rod are kept at zero]

① A rod of length 'l' with insulated sides is initially at a uniform temperature $f(x) = k(lx - x^2)$, $0 < x < l$. Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperature function $u(x, t)$

(or)

Solve the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ subject to the conditions

$$u(0, t) = 0, u(l, t) = 0 \text{ \& } u(x, 0) = k(lx - x^2), 0 < x < l$$

Solution :

Step 1 : The one dimensional heat equation is given by,

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Step 2 : The boundary conditions are,

(i) $u(0, t) = 0 \quad \forall t$

(ii) $u(l, t) = 0 \quad \forall t$

(iii) $u(x, 0) = k(lx - x^2), 0 < x < l$

Step 3 : The most suitable solution is given by,

$$u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-\alpha^2 p^2 t} \rightarrow \textcircled{1}$$

Step 4 : Applying condition (i) in $\textcircled{1}$,

$$u(0, t) = c_1 e^{-\alpha^2 p^2 t} = 0$$

Here $e^{-\alpha^2 p^2 t} \neq 0$ [\because It is defined for all t]

$$\therefore \boxed{c_1 = 0}$$

Subs $c_1 = 0$ in $\textcircled{1}$, we get

$$u(x, t) = c_2 \sin px e^{-\alpha^2 p^2 t} \rightarrow \textcircled{2}$$



Step 5: Applying condition (ii) in (2),

$$u(l, t) = c_2 \sin pl e^{-\alpha^2 p^2 t} = 0$$

$$e^{-\alpha^2 p^2 t} \neq 0 \quad [\because \text{It is defined for all } t]$$

$$c_2 \neq 0 \quad [\because c_1 = 0 \text{ we get a trivial solution}]$$

$$\therefore \sin pl = 0$$

$$\sin pl = \sin n\pi \quad [\because \sin n\pi = 0]$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Subs the value of p in (2),

$$u(x, t) = c_2 \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2 n^2 \pi^2 t / l^2}$$

Step 6:

The most general solution is,

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \rightarrow (3)$$

Step 7: Applying condition (iii) in (3),

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) = k(lx - x^2) \rightarrow (4)$$

Step 8: To find c_n :

Expand $f(x) = k(lx - x^2)$ as a half range sine

series in $(0, l)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \rightarrow (5)$$

$$\text{Where } b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$\text{From (4) \& (5), } c_n = b_n$$

$$\therefore c_n = \frac{2}{l} \int_0^l k(lx - x^2) \sin \frac{n\pi x}{l} dx$$



$$C_n = \frac{2K}{l} \left\{ (lx - x^2) \left(-\frac{l}{n\pi} \right) \cos \frac{n\pi x}{l} \right. \\ \left. + (l-2x) \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi x}{l} - \frac{2l^3}{n^3 \pi^3} \cos \frac{n\pi x}{l} \right\}_0^l$$

$$= -\frac{2K}{l} \left(\frac{2l^3}{n^3 \pi^3} \right) \left[\cos \left(\frac{n\pi x}{l} \right) \right]_0^l$$

$$= -\frac{4Kl^2}{n^3 \pi^3} [(-1)^n - 1]$$

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$$u = lx - x^2$$

$$u' = l - 2x, u'' = -2$$

$$v = \sin n\pi x / l$$

$$v_1 = -\cos n\pi x / l / \frac{n\pi}{l}$$

$$v_2 = -\sin n\pi x / l / \frac{n^2 \pi^2}{l^2}$$

$$v_3 = \cos n\pi x / l / \frac{n^3 \pi^3}{l^3}$$

$$C_n = \frac{4Kl^2}{n^3 \pi^3} [1 - (-1)^n]$$

Step 9: Subs the value of C_n in (3),

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4Kl^2}{n^3 \pi^3} [1 - (-1)^n] \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$



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TYPE 2 : Steady State Conditions and Zero Boundary

Conditions:

Steady State: The temperature does not vary w.r.t time 't' is called steady state.

Therefore, when steady state condition exists $u(x,t)$ becomes $u(x)$.

Steady State solution of one dimensional heat equation:

In unsteady state, one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

In steady state, $\frac{\partial u}{\partial t} = 0$

$$\therefore \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$$

\therefore The general solution is $u(x) = ax + b$ where a and b are arbitrary constants.

Problems:

- ① A rod 30 cm long has its end A and B kept at 20°C and 80°C respectively until steady state conditions prevail. Find the steady state temperature in the rod.

Solution:

The steady state one dimensional heat equation is,

$$\frac{d^2 u}{dx^2} = 0$$

The general solution is,

$$u(x) = ax + b \rightarrow \text{①}$$

The boundary conditions are,

(i) $u(0) = 20^\circ\text{C}$

(ii) $u(30) = 80^\circ\text{C}$

Applying Condition (i) in (1),

$$u(0) = b$$

$$\boxed{b = 20}$$

Applying Condition (ii) in (1),

$$u(30) = 30a + b$$

$$80 = 30a + 20$$

$$60 = 30a$$

$$\boxed{a = 2}$$

∴ The solution is, $\boxed{u(x) = 2x + 20}$

- (2) The ends A and B of a rod of length 10 cm long have their temperature kept 20°C and 70°C. Find the steady state temperature distribution on the rod.

Solution:

The steady state one dimensional heat equation is,

$$\frac{d^2u}{dx^2} = 0$$

The general solution is

$$u(x) = ax + b \rightarrow (1)$$

The boundary conditions are,

(i) $u(0) = 20^\circ\text{C}$

(ii) $u(10) = 70^\circ\text{C}$

Applying Condition (i) in (1),

$$u(0) = a(0) + b = 20$$

$$\boxed{b = 20}$$

Applying condition (ii) in (1),

$$u(10) = a(10) + b = 70$$

$$10a + 20 = 70$$

$$\boxed{a = 5}$$

Subs a & b in (1),

$$\boxed{u = 5x + 20}$$