Problems on Vibrating string with non-Zero initial velocity:

① A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating string gives each point a velocity $\lambda x(l-x)$ show that the displacement is

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi x}{l} \sin\frac{(2n-1)\pi at}{l}$$

Solution:

Step 1: The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

Step 2: The boundary and initial conditions are given by,

(i) $y(0,t) = 0$ for all $t > 0$

(ii) $y(l,t) = 0$ for all $t > 0$

(iii) $y(x,0) = 0$, $0 < x < l$ (initial displacement)

(iv) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = \lambda x(l-x)$, $0 < x < l$. (initial velocity)

Step 3: The suitable solution which satisfies the boundary Conditions is given by,

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \quad \longrightarrow ①$$

Step 4: Applying condition(i) in ①, we get,

$$y(0,t) = c_1 (c_3 \cos pat + c_4 \sin pat) = 0.$$

Here, $c_3 \cos pat + c_4 \sin pat \neq 0$ [$\because$ it is defined for all t]

Therefore, we get $\boxed{c_1 = 0}$

Subs $c_1 = 0$ in (1), we get

$$y(x,t) = c_2 \sin px \,(c_3 \cos pat + c_4 \sin pat) \longrightarrow (2)$$

Step 5: Applying condition (ii) in eqn (2), we get

$$y(l,t) = c_2 \sin pl \,(c_3 \cos pat + c_4 \sin pat) = 0.$$

Here $(c_3 \cos pat + c_4 \sin pat) \neq 0$ [$\because$ It is defined for all t]

Therefore, either $c_2 = 0$ or $\sin pl = 0$.

Suppose, we take $c_2 = 0$ and already we have $c_1 = 0$ then we get a trivial solution.

Therefore $c_2 \neq 0$ & $\sin pl = 0$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}, \text{ n is an integer}$$

Subs $p = \frac{n\pi}{l}$ in eqn (2), we get

$$y(x,t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}\right) \longrightarrow (3)$$

Step 6:

Applying condition (iii) in eqn (3), we get,

$$y(x,0) = c_2 \sin \frac{n\pi x}{l} \cdot c_3 = 0$$

$$\Rightarrow c_2 c_3 \sin \frac{n\pi x}{l} = 0$$

Here, $\sin \frac{n\pi x}{l} \neq 0$ [$\because$ It is defined for all x]

$$c_2 \neq 0$$

$$\therefore \boxed{c_3 = 0}$$

Subs $c_3 = 0$ in eqn (3), we get

$$y(x,t) = c_2 c_4 \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l}$$

$$= C_n \sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l} \longrightarrow (4) \quad \text{where } C_n = C_2 C_4$$

Step 7:

The most general solution is,

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l} \longrightarrow (5)$$

Step 8:

Diff (5) w.r.t 't',

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} C_n \left(\sin\frac{n\pi x}{l}\right)\left(\frac{n\pi a}{l}\right)\cos\frac{n\pi a t}{l}$$

Applying condition (iv), we get,

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi a}{l}\right) = \lambda x(l-x)$$

$$\sum_{n=1}^{\infty}\left[C_n\cdot\frac{n\pi a}{l}\right]\sin\frac{n\pi x}{l} = \lambda x(l-x)$$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \lambda x(l-x) \longrightarrow (6)$$

where $B_n = C_n\cdot\frac{n\pi a}{l}$.

Step 9:

To find B_n: Expand $\lambda x(l-x)$ in a half-range Fourier Sine series in the interval $(0,l)$.

$$\lambda x(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \longrightarrow (7)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

From (6) & (7), we get $\boxed{b_n = B_n}$

$$\therefore B_n = \frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\int_0^l (xl-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\left\{-(xl-x^2)\frac{l}{n\pi}\cos\frac{n\pi x}{l} + (l-2x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} - 2\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right\}_0^l$$

$u = xl - x^2,\ u' = l - 2x$
$u'' = -2$
$v = \sin n\pi x/l$
$v_1 = -\cos\frac{n\pi x}{l}\Big/\frac{n\pi}{l}$
$v_2 = -\sin\frac{n\pi x}{l}\Big/\frac{n^2\pi^2}{l^2}$
$v_3 = \cos\frac{n\pi x}{l}\Big/\frac{n^3\pi^3}{l^3}$

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$$B_n = \frac{2\lambda}{l}\left(\frac{-2l^3}{n^3\pi^3}\right)[\cos n\pi - \cos 0]$$

$$B_n = \frac{4\lambda l^2}{n^3\pi^3}[1-(-1)^n]$$

$$C_n\left(\frac{n\pi a}{l}\right) = \frac{4\lambda l^2}{n^3\pi^3}[1-(-1)^n]$$

$$C_n = \left(\frac{1}{n\pi a}\right)\left(\frac{4\lambda l^3}{n^3\pi^3}\right)[1-(-1)^n]$$

$$C_n = \frac{4\lambda l^3}{an^4\pi^4}[1-(-1)^n]$$

i.e., $C_n = \begin{cases} 0 & , n \text{ is even} \\ \dfrac{8\lambda l^3}{an^4\pi^4} & , n \text{ is odd.} \end{cases}$

Step 10:

Subs the value of C_n in eqn 5, we get

$$y(x,t) = \sum_{n=odd}^{\infty} \frac{8\lambda l^3}{an^4\pi^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4}\sum_{n=odd}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= \frac{8\lambda l^3}{a\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{l}\sin\frac{(2n-1)\pi at}{l}$$

2. If a string of length l is initially at rest in its equilibrium position and each of its points is given the velocity $V_0 \sin^3\frac{\pi x}{l}$, $0<x<l$, determine the displacement of a point distant x from one end at time 't'.

Solution:

Step 1: The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

Step 2 : From the given problem, we have the following initial and boundary conditions.

(i) $y(0,t) = 0 \quad \forall\ t$

(ii) $y(l,t) = 0 \quad \forall\ t$

(iii) $y(x,0) = 0 \quad 0 < x < l$ (initial displacement)

(iv) $\frac{\partial y}{\partial t}(x,0) = V_0 \sin^3\left(\frac{\pi x}{l}\right)$, $0 < x < l$ (initial velocity)

Step 3 : The suitable solution which satisfies the initial and boundary conditions is,

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \longrightarrow (1)$$

Step 4: Applying condition (i) in (1), we get

$$y(0,t) = (c_1 \cos p0 + c_2 \sin p0)(c_3 \cos pat + c_4 \sin pat)$$

$$0 = c_1 (c_3 \cos pat + c_4 \sin pat)$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0$ ($\because$ it is defined for all t)

$\therefore \boxed{c_1 = 0}$

subs $c_1 = 0$ in (1), we get

$$y(x,t) = c_2 \sin px\,(c_3 \cos pat + c_4 \sin pat) \longrightarrow (2)$$

Step 5 : Applying condition (ii) in (2), we get

$$y(l,t) = c_2 \sin pl\,(c_3 \cos pat + c_4 \sin pat)$$

$$0 = c_2 \sin pl\,(c_3 \cos pat + c_4 \sin pat)$$

Here $c_3 \cos pat + c_4 \sin pat \neq 0$ [$\because$ it is defined for all t]

$c_2 \neq 0$ ($\because$ $c_1 = 0$, we get a trivial soln)

$\therefore \sin pl = 0$

$\sin pl = \sin n\pi$

$pl = n\pi$

$$\boxed{p = \frac{n\pi}{l}}$$

Subs $p = \frac{n\pi}{l}$ in (2), we get,

$$y(x,t) = c_2 \sin \frac{n\pi x}{l} \left[c_3 \cos\left(\frac{n\pi a t}{l}\right) + c_4 \sin\left(\frac{n\pi a t}{l}\right)\right] \longrightarrow (3)$$

Step 6 :

Applying (iii) in (3),

$$y(x,0) = c_2 \sin\left(\frac{n\pi x}{l}\right) c_3 = 0$$

$$c_2 c_3 \sin\left(\frac{n\pi x}{l}\right) = 0$$

Here $\sin\left(\frac{n\pi x}{l}\right) \neq 0$ [$\because$ it is defined for all x]

$c_2 \neq 0$ [$\because c_1 = 0$, we get a trivial solution]

$\therefore \boxed{c_3 = 0}$

Subs $c_3 = 0$ in (3), we get

$$y(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right) c_4 \sin\left(\frac{n\pi a t}{l}\right)$$

i.e., $y(x,t) = c_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi a t}{l}\right)$ where $c_n = c_2 c_4$

Step 7 :

The most general solution is given by,

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi a t}{l}\right) \longrightarrow (4)$$

Step 8 :

Before applying condition (iv), let us find $\frac{\partial y}{\partial t}(x,t)$.

Diff (4) partially w.r.t 't', we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi a}{l}\right) \cos\left(\frac{n\pi a t}{l}\right)$$

Putting $t=0$, we get

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \cdot \left(\frac{n\pi a}{l}\right)$$

$$V_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} c_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \rightarrow (5)$$

Step 9 :

We know that,

$$\sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$$

$$\sin^3\left(\frac{\pi x}{l}\right) = \frac{1}{4}\left[3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right]$$

$$V_0 \sin^3\left(\frac{\pi x}{l}\right) = \frac{V_0}{4}\left[3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right] \rightarrow (6)$$

From (5) & (6),

$$\sum_{n=1}^{\infty} c_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) = \frac{V_0}{4}\left[3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right]$$

$$c_1 \sin\left(\frac{\pi a}{l}\right)\sin\left(\frac{\pi x}{l}\right) + c_2 \sin\left(\frac{2\pi a}{l}\right)\sin\left(\frac{2\pi x}{l}\right) + c_3\left(\frac{3\pi a}{l}\right)\sin\left(\frac{3\pi x}{l}\right) + \cdots$$

$$= \frac{V_0}{4}\left[3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right]$$

Equating the coefficients on both sides, we get,

$$c_1\left(\frac{\pi a}{l}\right) = \frac{3V_0}{4}, \quad c_2 = 0, \quad c_3\left(\frac{3\pi a}{l}\right) = -\frac{V_0}{4},$$

$$c_4 = 0, \quad c_5 = 0, \quad \cdots\cdots$$

i.e., $c_1 = \dfrac{3 l V_0}{4\pi a}, \quad c_2 = 0, \quad c_3 = -\dfrac{l V_0}{12\pi a}, \quad c_4 = 0, \cdots\cdots \rightarrow (7)$

Step 10 :

Subs (7) in (4), we get

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi a t}{l}\right)$$

i.e., $y(x,t) = \dfrac{3lV_0}{4\pi a}\sin\left(\dfrac{\pi x}{l}\right)\sin\left(\dfrac{\pi a t}{l}\right) - \dfrac{lV_0}{12\pi a}\sin\left(\dfrac{3\pi x}{l}\right)\sin\left(\dfrac{3\pi a t}{l}\right)$