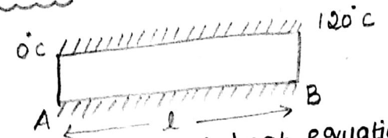
Type II : Problems :

(1) A rod of length 'l' has its ends A and B kept at $0^\circ c$ and $120^\circ c$ respectively until steady state conditions prevails. If the temperature at B reduced to $0^\circ c$ and kept so while that of A is maintained, find the temperature distribution in the rod.

Solution :

In Steady State :



The one dimensional heat equation is,

$$\frac{d^2u}{dx^2} = 0$$

The boundary conditions are,

(i) $u(0) = 0$

(ii) $u(l) = 120$

The steady state solution is,

$$u(x) = ax + b \longrightarrow (1)$$

Applying condition (i) in (1),

$$u(0) = a(0) + b = 0$$

$$\boxed{b = 0}$$

Applying condition (ii) in (1),

$$u(l) = al + b = 120$$

$$al = 120$$

$$\boxed{a = \frac{120}{l}}$$

subs a and b in (1),

$$u(x) = \frac{120x}{l}, \quad 0 \le x \le l.$$

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If the temperature at B is reduced to 0°c, then the temperature distribution changes from steady state to unsteady state.

Step 1 : The one dimensional heat equation is,

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Step 2 : The boundary conditions are,

(i) $u(0,t) = 0 \quad \forall\ t \geq 0$

(ii) $u(l,t) = 0 \quad \forall\ t \geq 0$

(iii) $u(x,0) = \frac{120x}{l}, \quad 0 < x < l.$

Step 3 : The correct solution is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, e^{-\alpha^2 p^2 t} \rightarrow (1)$$

Step 4 : Applying (i) in (1),

$$u(0,t) = c_1 e^{-\alpha^2 p^2 t} = 0$$

Here $e^{-\alpha^2 p^2 t} \neq 0$ [$\because$ It is defined for all t]

$$\therefore c_1 = 0$$

$$(1) \Rightarrow u(x,t) = c_2 \sin px\, e^{-\alpha^2 p^2 t} \rightarrow (2)$$

Step 5 : Applying (ii) in (2),

$$u(l,t) = c_2 \sin pl\, e^{-\alpha^2 p^2 t} = 0$$

Here $e^{-\alpha^2 p^2 t} \neq 0$ [$\because$ It is defined for all t]

$c_2 \neq 0$ [$\because c_1 = 0$ we get a trivial solution]

$$\therefore \sin pl = 0 = \sin n\pi$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

$$(2) \Rightarrow u(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2 n^2 \pi^2 t / l^2}$$

$$u(x,t) = c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2 n^2 \pi^2 t / l^2}$$

Step 6 : The most general solution is,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2 n^2 \pi^2 t/l^2} \longrightarrow (3)$$

Step 7 : Applying (iii) in (3),

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) = \frac{120x}{l} \longrightarrow (4)$$

Step 8 : To find c_n :

Expand $f(x) = \frac{120x}{l}$ as a half range Fourier sine series in $(0, l)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \longrightarrow (5)$$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

From (4) & (5), $\boxed{b_n = c_n}$

$$c_n = \frac{2}{l} \int_0^l \frac{120x}{l} \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{240}{l^2} \left[x\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{n\pi/l}\right) + \frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2} \right]_0^l$$

$$= \frac{240}{l^2} \left[-l \cdot \frac{l}{n\pi} \cos n\pi \right]$$

$$c_n = -\frac{240}{n\pi} (-1)^n$$

$$\boxed{c_n = \frac{240}{n\pi} (-1)^{n+1}}$$

Step 9 : Subs the value of c_n in (3),

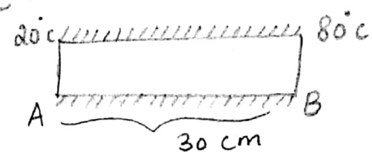
$$u(x,t) = \sum_{n=1}^{\infty} \frac{240}{n\pi} (-1)^{n+1} \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2 n^2 \pi^2 t/l^2}$$

Both ends are change to zero temp. (30)

(2) A rod 30 cm long has its ends A and B kept at $20^\circ c$ and $80^\circ c$ respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to $0^\circ c$ and kept so. Find the resulting temperature function $u(x,t)$ taking $x=0$ at A.

Solution :

In Steady state :



The one dimensional heat equation is,

$$\frac{d^2u}{dx^2} = 0$$

The boundary conditions are,

(i) $u(0) = 20^\circ c$

(ii) $u(30) = 80^\circ c$

The steady state solution is,

$$u(x) = ax + b \longrightarrow (1)$$

Applying Condition (i) in (1),

$$u(0) = a(0) + b = 20$$

$$\boxed{b = 20}$$

Applying Condition (ii) in (1),

$$u(30) = a(30) + b = 80$$

$$30a + 20 = 80$$

$$30a = 60$$

$$\boxed{a = 2}$$

Subs a and b in (1),

$$\boxed{u(x) = 2x + 20}$$

If the temperature at each end is reduced to $0^\circ c$, then the temperature distribution changes from steady state to unsteady state.

Step 1 : The one dimensional heat equation is,

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Step 2 : The boundary conditions are,

(i) $u(0,t) = 0 \quad \forall\, t \geq 0$

(ii) $u(30,t) = 0 \quad \forall\, t \geq 0$

(iii) $u(x,0) = 2x + 20 \,, \quad 0 < x < 30$

Step 3 : The correct solution is,

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, e^{-\alpha^2 p^2 t} \longrightarrow (1)$$

Step 4 : Applying condition (i) in (1),

$$u(0,t) = c_1 e^{-\alpha^2 p^2 t} = 0$$

Here $e^{-\alpha^2 p^2 t} \neq 0$ [$\because$ It is defined for all t]

$$\therefore \boxed{c_1 = 0}$$

$$(1) \Rightarrow u(x,t) = c_2 \sin px\, e^{-\alpha^2 p^2 t} \longrightarrow (2)$$

Step 5 : Applying condition (ii) in (2),

$$u(30,t) = c_2 \sin 30p\, e^{-\alpha^2 p^2 t} = 0$$

Here $e^{-\alpha^2 p^2 t} \neq 0$ [$\because$ It is defined for all t]

$c_2 \neq 0$ [$\because$ $c_1 = 0$ we get a trivial solution]

$$\sin 30p = 0 = \sin n\pi$$

$$30p = n\pi$$

$$\boxed{p = \frac{n\pi}{30}}$$

$$(2) \Rightarrow u(x,t) = c_2 \sin\left(\frac{n\pi x}{30}\right) e^{-\alpha^2 n^2 \pi^2 t/30^2}$$

$$u(x,t) = c_n \sin\left(\frac{n\pi x}{30}\right) e^{-\alpha^2 n^2 \pi^2 t/30^2}$$

Step 6 : The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{30}\right) e^{-\alpha^2 n^2 \pi^2 t/30^2} \longrightarrow (3)$$

Step 7: Applying condition (iii) in 3,

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{30}\right) = 2x+20 \longrightarrow 4$$

Step 8: To find c_n:

Expand $f(x) = 2x+20$ as a half range Fourier sine series in $(0,30)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{30}\right) \longrightarrow 5$$

where $b_n = \frac{2}{30}\int_0^{30} f(x) \sin\left(\frac{n\pi x}{30}\right) dx$

From 4 & 5, $\boxed{b_n = c_n}$

$$c_n = \frac{2}{30}\int_0^{30} (2x+20) \sin\left(\frac{n\pi x}{30}\right) dx$$

$$= \frac{2}{30}\left\{(2x+20)\left(-\frac{\cos(n\pi x/30)}{n\pi/30}\right) + 2\cdot\frac{30^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{30}\right)\right\}_0^{30}$$

$u = 2x+20$
$u' = 2$
$v = \sin(n\pi x/30)$
$v_1 = -\frac{\cos(n\pi x/30)}{n\pi/30}$
$v_2 = -\frac{\sin(n\pi x/30)}{n^2\pi^2/30^2}$

$$= \frac{2}{30}\left\{(60+20)\left(\frac{-30}{n\pi}\right)\cos n\pi + 20\left(\frac{30}{n\pi}\right)\right\}$$

$$= \frac{2}{30}\cdot\frac{30}{n\pi}\cdot 20\left\{1 - 4\cos n\pi\right\}$$

$$\boxed{c_n = \frac{40}{n\pi}\left[1-4(-1)^n\right]}$$

Step 9: Subs the value of c_n in 3,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left[1-4(-1)^n\right] \sin\left(\frac{n\pi x}{30}\right) e^{-\alpha^2 n^2\pi^2 t/30^2}$$