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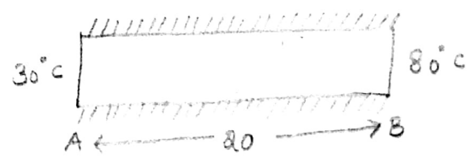
TYPE III : Problems on Steady State conditions and non-zero boundary conditions (or) Non-zero temperatures at the ends of the bar, both in steady state and unsteady state :

(1) Two ends A and B of a rod of length 20 cms have their temperatures at 30°c and 80°c respectively until steady state conditions prevail. Then the temperatures at the ends A and B are changed to 40°c and 60°c respectively. Find the temperature distribution of the rod at any time 't'.

Solution :

Steady State I :



The steady state solution is

$$u = ax + b$$

(i) $u(0) = 30°c$

(ii) $u(20) = 80°c$

$$u(0) = a(0) + b = 30$$

$$\boxed{b = 30}$$

$$u(20) = 20a + b = 80$$

$$20a + 30 = 80$$

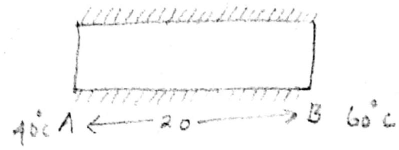
$$20a = 50$$

$$\boxed{a = 5/2}$$

$\therefore \boxed{u(x) = \frac{5}{2}x + 30}$ which is the initial temperature of the rod. i.e.,

$$u(x,0) = \frac{5}{2}x + 30$$

Steady state 2 :



The solution is , $u(x) = ax + b \longrightarrow (1)$

(i) $u(0) = 40°c$

(ii) $u(20) = 60°c$

App (i) in (1) $\Rightarrow u(0) = a(0) + b = 40$

$$\boxed{b = 40}$$

App (ii) in (1) $\Rightarrow$ $u(20) = 20a + b = 60$

$$20a + 40 = 60$$

$$\boxed{a = 1}$$

$\therefore$ $u(x) = x + 40$

which is the transient state temperature of the rod.

i.e., $u_t(x,t) = x + 40$.

Step 1 : The one-dimensional heat equation is,

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Step 2 : The boundary conditions are,

(i) $u(0,t) = 40$

(ii) $u(20,t) = 60$

(iii) $u(x,0) = \frac{5}{2}x + 30$

Step 3 : The correct solution is ,

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, e^{-\alpha^2 p^2 t} + u_t$$

$$u(x,t) = x + 40 + (c_1 \cos px + c_2 \sin px)\, e^{-\alpha^2 p^2 t} \longrightarrow (1)$$

Step 4 : Applying condition (i) in (1),

$$u(0,t) = 40 + c_1 e^{-\alpha^2 p^2 t}$$

$$40 = 40 + c_1 e^{-\alpha^2 p^2 t}$$

$$0 = c_1 e^{-\alpha^2 p^2 t}$$

(33)

Here $e^{-\alpha^2 p^2 t} \neq 0$ [$\because$ It is defined for all t]

$\therefore c_1 = 0$

$$(1) \Rightarrow u(x,t) = x + 40 + c_2 \sin px \, e^{-\alpha^2 p^2 t} \longrightarrow (2)$$

Step 5 : Applying condition (ii) in (2).

$$u(20,t) = 20 + 40 + c_2 \sin 20p \, e^{-\alpha^2 p^2 t}$$

$$60 = 60 + c_2 \sin 20p \, e^{-\alpha^2 p^2 t}$$

$$0 = c_2 \sin 20p \, e^{-\alpha^2 p^2 t}$$

Here $e^{-\alpha^2 p^2 t} \neq 0$ [$\because$ It is defined for all t]

$c_2 \neq 0$ [$\because c_1 = 0$ we get a trivial solution]

$$\therefore \sin 20p = 0$$

$$\sin 20p = \sin n\pi$$

$$20p = n\pi$$

$$\boxed{p = \frac{n\pi}{20}}$$

$$(2) \Rightarrow u(x,t) = x + 40 + c_2 \sin\left(\frac{n\pi x}{20}\right) e^{-\alpha^2 n^2 \pi^2 t/20^2}$$

$$u(x,t) = x + 40 + C_n \sin\left(\frac{n\pi x}{20}\right) e^{-\alpha^2 n^2 \pi^2 t/400}$$

Step 6 : The most general solution is,

$$u(x,t) = x + 40 + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{20}\right) e^{-\alpha^2 n^2 \pi^2 t/400} \longrightarrow (3)$$

Step 7 : Applying condition (iii) in (3),

$$u(x,0) = x + 40 + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{20}\right)$$

$$\frac{5}{2}x + 30 = x + 40 + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{20}\right)$$

$$\frac{5}{2}x - x + 30 - 40 = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{20}\right)$$

$$\frac{3}{2}x - 10 = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{20}\right) \longrightarrow (4)$$