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APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Initial Conditions:

The conditions which are defined at time $t=0$ are called initial conditions.

Boundary Conditions:

The conditions which are defined at the boundary of the region or interval are called boundary conditions.

Boundary value problems:

The partial differential equations which satisfy certain initial and boundary conditions are called boundary value problems

Classification and Characteristics of a P.D.E:

The general form of Second order PDE is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \longrightarrow (1)$$

which can also be written as,

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + Fu = G$$

where A, B, C, D, E, F & G are constants (or) functions of x and y only.

Classification:

The P.D.E (1) is said to be.

(i) Elliptic if $B^2 - 4AC < 0$

(ii) Parabolic if $B^2 - 4AC = 0$

(iii) Hyperbolic if $B^2 - 4AC > 0$

PROBLEMS :

① Classify $u_{xx} + 2u_{xy} + u_{yy} + u_x - u_y = 0$

Solution:

Here $A = 1$, $B = 2$, $C = 1$

$B^2 - 4AC = 4 - 4 = 0$

$\therefore$ The given PDE is parabolic in nature.

② Classify $2f_{xx} + f_{xy} - f_{yy} + f_x + 3f_y = 0$.

Solution:

Here $A = 2$, $B = 1$, $C = -1$

$B^2 - 4AC = 1 - 4(2)(-1) = 1 + 8 = 9 > 0$

$\therefore$ The given PDE is hyperbolic in nature.

③ Classify the PDE $3u_{xx} + 2u_{xy} + 5u_{yy} + xu_y = 0$.

Solution:

Here $A = 3$, $B = 2$, $C = 5$

$B^2 - 4AC = 4 - 4(3)(5) = 4 - 60 = -56 < 0$

$\therefore$ The given PDE is elliptic in nature.

④ Classify $(1+x)^2 u_{xx} - 4x u_{xy} + u_{yy} = x$

Solution:

Here $A = (1+x)^2$, $B = -4x$, $C = 1$

$$
\begin{aligned}
B^2 - 4AC &= (-4x)^2 - 4(1+x)^2 \\
&= 16x^2 - 4(1 + 2x + x^2) \\
&= 16x^2 - 4 - 8x - 4x^2 \\
&= 12x^2 - 8x - 4 \\
&= 4(3x^2 - 2x - 1)
\end{aligned}
$$

If $x = 1$, $B^2 - 4AC = 0$, then the PDE is parabolic at $x=1$

If x is positive i.e., $x > 0$ then $B^2 - 4AC > 0$, the PDE is hyperbolic

If x is negative i.e., $x < 0$ then $B^2 - 4AC > 0$, the PDE is hyperbolic

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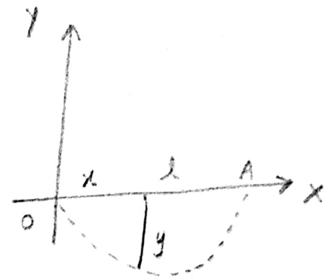
ONE DIMENSIONAL WAVE EQUATION (OR) EQUATION OF VIBRATING STRING :

Consider an elastic string, tightly stretched between two points O and A.



Let O be the origin and OA as X-axis.

Give a small displacement to the string, perpendicular to its length.

Let y be the displacement at any point, at any time.

Then the equation of the vibrating string is given by,

$$\boxed{\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}}$$

which is also known as one-dimensional wave equation

where $a^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{mass per unit length of the string}}$

NOTE :

$y(x,t)$ is the displacement of the string at a distance 'x' from one end at time 't'.

Various Solutions of wave equations :

The possible solutions of wave equations are,

(1) $y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{apt} + c_4 e^{-apt})$

(2) $y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat)$

(3) $y(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

Here the most suitable solution is given by

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat)$$