CONVOLUTION THEOREM:

If $F[f(x)]$ & $F[g(x)]$ are the Fourier transform of $f(x)$ & $g(x)$ respectively then the Fourier transform of the convolution of $f(x)$ & $g(x)$ is the product of their Fourier transforms.

(i.e.) $F[f(x) * g(x)] = F(s) \cdot G(s)$

$$= F[f(x)] F[g(x)]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx * \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{isx} dx.$$

CONVOLUTION OF ANY TWO FUNCTIONS:

$f(x)$ and $g(x)$ in Fourier transforms is denoted by $(f * g)(x) = f(x) * g(x)$ and is defined by,

$$(f * g)(x) = f(x) * g(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot g(x-t)\, dt.$$

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx\, dx = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right].$$

$$G_c(s) = G_c[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-bx} \cos sx\, dx = \sqrt{\frac{2}{\pi}} \left[\frac{b}{b^2+s^2}\right]$$

1) Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using transforms (or)

Find the Fourier cosine transform of $f(x) = e^{-ax}$ & $g(x) = e^{-bx}$ & evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ (or)

Evaluate parseval's Identity $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$

Soln:

WKT $F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right]$

$G_c(s) = G_c[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \cos sx \, dx. = \sqrt{\frac{2}{\pi}} \left[\frac{b}{b^2+s^2}\right]$

parseval's Identity:

$$\int_0^\infty f(x) \cdot g(x) dx = \int_0^\infty F_c(s) \cdot G_c(s) ds$$

$$= \int_0^\infty F_c[f(x)] \cdot F_c[g(x)] ds$$

Here $f(x) = e^{-ax}$, $g(x) = e^{-bx}$

$$\int_0^\infty e^{-ax} e^{-bx} dx = \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right] \cdot \sqrt{\frac{2}{\pi}} \left[\frac{b}{b^2+s^2}\right] ds .$$

$$\int_0^\infty e^{-(a+b)x}\,dx = \frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+s^2)(b^2+s^2)}\,ds$$

$$\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)}$$

$$\frac{e^{-\infty}-e^{0}}{-(a+b)} = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)}$$

$$\frac{1}{a+b}\cdot\frac{\pi}{2ab} = \int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)}$$

put $s=x$.

$$\frac{\pi}{2ab(a+b)} = \int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

2) Evaluate $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$ using transforms.

Soln:

WKT $F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]$

$G_s(s) = F_s[g(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{s}{b^2+s^2}\right]$

Parseval's identity:

$$\int_0^\infty f(x)\,g(x)\,dx = \int_0^\infty F_s(s)\,G_s(s)\,ds$$

Here $f(x)=e^{-ax}$, $g(x)=e^{-bx}$

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_s(s)\, G_s(s)\, ds$$

Here $f(x) = e^{-ax}$, $g(x) = e^{-bx}$

$$\int_0^\infty e^{-ax} \cdot e^{-bx}\, dx = \int_0^\infty F_s[f(x)]\, F_s[g(x)]\, ds$$

$$\int_0^\infty e^{-(a+b)x}\, dx = \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right] \cdot \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+b^2}\right] ds$$

$$\left.\frac{e^{-(a+b)}}{-(a+b)}\right]_0^\infty = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds$$

$$\frac{1}{a+b} \cdot \frac{\pi}{2} = \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds$$

put $s = x$

$$\frac{\pi}{2(a+b)} = \int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx.$$