SINE TRANSFORM :

The Fourier sine transform of a function $f(x), 0<x<\infty$ is defined as $F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

The Inverse Fourier sine transform of $F_s(s)$ is defined as $f(x) = F^{-1}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$.

Parseval's Identity: If $F_s(s)$ is the Fourier transform of $f(x)$ then $\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_s(s)]^2 ds$.

COSINE TRANSFORM :

The Fourier cosine transform of a function $f(x), 0<x<\infty$ is defined as $F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$.

The Inverse Fourier cosine transform of $F_c(s)$ is defined as $f(x) = F^{-1}[F_c(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$.

If $F_c(s)$ is the Fourier transform of $f(x)$ then parseval's Identity is $\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_c(s)]^2 ds$

4) Find the Fourier cosine transform of $f(x) = \begin{cases} \cos x, & \text{if } 0<x<a \\ 0, & \text{if } x \geq a \end{cases}$

Soln:

WKT $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \cos x \cdot \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \frac{1}{2}\left[\cos(s+1)x + \cos(s-1)x\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^a \left[\cos(s+1)x + \cos(s-1)x\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1}\right]$$

5) Find the Fourier sine & cosine transform of e^{-ax} and deduce that inverse fourier transform & parseval's identity

Soln:

Sine transform:

WKT $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$$

Inverse Transform:

WKT $f(x) = F^{-1}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \sin sx \, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds$$

$$\frac{\pi}{2} e^{-ax} = \int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds$$

Parseval's Identity:

WKT $\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_s(s)]^2 ds$

$$\int_0^\infty (e^{-ax})^2 dx = \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]\right]^2 ds$$

$$\int_0^\infty e^{-2ax} dx = \frac{2}{\pi} \int_0^\infty \left(\frac{s}{s^2+a^2}\right)^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{1}{2a} = \frac{2}{\pi} \int_0^\infty \left[\frac{s}{s^2+a^2}\right]^2 ds$$

$$\frac{\pi}{4a} = \int_0^\infty \left[\frac{s}{s^2+a^2}\right]^2 ds$$

Cosine Transform:

WKT $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$

$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$

Inverse Transform:

WKT $f(x) = F^{-1}[F_c(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$

$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right] \cos sx \, ds$

$e^{-ax} = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{a}{s^2+a^2} \cdot \cos sx \, ds$

put $x = 0$ $\quad 1 = \frac{2}{\pi} \int_0^\infty \frac{a}{s^2+a^2} \cos s(0) \, ds$

$\frac{\pi}{2} = \int_0^\infty \frac{a}{s^2+a^2} \, ds .$

Parseval's Identity:

WKT $\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_c(s)]^2 ds$

$$\int_0^\infty [e^{-ax}]^2 dx = \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}\right]^2 ds$$

$$\int_0^\infty e^{-2ax} dx = \frac{2}{\pi} \int_0^\infty \left(\frac{a}{s^2+a^2}\right]^2 ds$$

$$\left.\frac{e^{-2ax}}{-2a}\right]_0^\infty : \frac{1}{2a} = \frac{2}{\pi} \int_0^\infty \left[\frac{a}{s^2+a^2}\right]^2 ds$$

$$\frac{\pi}{4a} = \int_0^\infty \left[\frac{a}{s^2+a^2}\right]^2 ds.$$