FOURIER SINE & COSINE TRANSFORM WITH PARSEVAL'S IDENTITY:

SINE TRANSFORM:

The Fourier sine transform of a function $f(x), 0<x<\infty$ is defined as $F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

The Inverse Fourier sine transform of $F_s(s)$ is defined as $f(x) = F^{-1}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$.

Parseval's Identity: If $F_s(s)$ is the Fourier transform of $f(x)$ then $\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_s(s)]^2 ds$.

NOTE: $F_s(s)$ and $F^{-1}[F_s(s)]$ is called Fourier sine transform pair.

COSINE TRANSFORM:

The Fourier cosine transform of a function $f(x), 0<x<\infty$ is defined as $F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

The Inverse Fourier cosine transform of $F_c(s)$ is defined as $f(x) = F^{-1}[F_c(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$.

If $F_c(s)$ is the Fourier transform of $f(x)$ then Parseval's Identity is $\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_c(s)]^2 ds$

NOTE: $F_c(s)$ and $F^{-1}[F_c(s)]$ is called Fourier cosine transform pair.

2> Find the Fourier sine transform of $1/x$.

Soln: WKT $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x} \, dx$$

putting $\theta = sx \Rightarrow d\theta = s\,dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta/s} \, d\theta/s$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta} \, d\theta \qquad \left[\because \int_0^\infty \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2}$$

$$= \sqrt{\pi/2}$$

1> Find the Fourier sine transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Soln: WKT $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos sx}{s}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s}\right]$$

3) Find the Fourier cosine transform of $2e^{-3x} + 3e^{-2x}$.

Soln:

WKT $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left(2e^{-3x} + 3e^{-2x}\right) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[2\left[\frac{3}{s^2+9}\right] + 3\left[\frac{2}{s^2+4}\right] \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{6}{s^2+9} + \frac{6}{s^2+4} \right]$$