Cayley - Hamilton Theorem :—

Every square matrix satisfies its own char. eqn

Applications :-

(i) To calculate the +ve integral powers &
(ii) To calculate the inverse of a square matrix A.

(i) verify that $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ satisfies its own char eqn. & find A^4.

soln: The char. eqn. is $\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = 0 ; S_2 = -5$

$\therefore$ the char. eqn. is $\lambda^2 - 5 = 0$.

To prove: $A^2 - 0A + 5I = 0$. (or) $A^2 - 5I = 0$

Now $A^2 - 5I = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$= 0$

$\therefore$ The gn. matrix satisfies its own char. eqn.

2) Verify Cayley-Hamilton theorem find A^4 and A^{-1} when $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Soln: The char. Eqn. is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Here $S_1 = 6$; $S_2 = 8$; $S_3 = 3$

$\therefore$ the char. Eqn. is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To prove: CHT (i.e) $A^3 - 6A^2 + 8A - 3I = 0$ —(1)

Now $A^3 - 6A^2 + 8A - 3I$

$$= \underset{A^2\times A}{\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}^3} - 6\underset{A\times A}{\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}^2} + 8\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0$

$\therefore$ the gn. matrix satisfies its own char. Eqn.

To find: A^4

From (1) $A^3 = 6A^2 - 8A + 3I$

$$A \times A^3 = A \times [6A^2 - 8A + 3I]$$

$$A^4 = 6A^3 - 8A^2 + 3A$$

$$= 6[6A^2 - 8A + 3I] - 8A^2 + 3A$$

$$= 36A^2 - 48A + 18I - 8A^2 + 3A$$

$$= 28A^2 - 45A + 18I$$

$$= 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1}:

From (1) $A^3 - 6A^2 + 8A - 3I = 0$

'$\times$' A^{-1} we get, $A^2 - 6A + 8 - 3A^{-1} = 0$

$$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$$

$$A^{-1} = \frac{1}{3}\left\{\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

3) use cayley-Hamilton theorem to find the value of the matrix eqn. by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ if the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

<u>soln:</u> The char. Eqn. is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 5$; $S_2 = 7$; $S_3 = 3$

$\therefore$ The char. Eqn. is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By CHT we get $A^3-5A^2+7A-3I=0$.

To find the value of $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

$$\begin{array}{r|l}
 & A^5+A \\
\hline
A^3-5A^2+7A-3I & A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I \\
 & A^8-5A^7+7A^6-3A^5 \\
\cline{2-2}
 & A^4-5A^3+8A^2-2A \\
 & A^4-5A^4+7A^2-3A \\
\cline{2-2}
 & A^2+A+I.
\end{array}$$

$\therefore A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

$=(A^3-5A^2+7A-3I)(A^5+A)+(A^2+A+I)$

$= 0\,(A^5+A)+(A^2+A+I)$ $\quad[\because \text{CHT}]$

$=A^2+A+I$

Now $A^2+A+I = \begin{bmatrix}2&1&1\\0&1&0\\0&1&2\end{bmatrix}^2+\begin{bmatrix}2&1&1\\0&1&0\\0&1&2\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$=\begin{bmatrix}8&5&5\\0&3&0\\5&5&8\end{bmatrix}$

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